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ABSTRACT

Unit 17 in the SMSG secondary school mathematics series is a student text covering the following topics: number systems, coordinate geometry in the plane, the function concept and the linear function, quadratic functions and equations, complex number systems, equations of the first and second degree in two variables, systems of equations in two variables, and systems of first degree equations in three variables. (DT)

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STUDENT'S TEXT

UNIT NO. 17

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INTERMEDIATE MATHEMATICS

PART I

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SCHOOL MATHEMATICS STUDY GROUP
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School Mathematics Study Group

Intermediate Mathematics

Unit 17

Intermediate Mathematics

Student's Text, Part I

Prepared under the supervision of
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FOREWORD

The increasing contribution of mathematics to the culture of the modern world, as well as its importance as a vital part of scientific and humanistic education, has made it essential that the mathematics in our schools be both well selected and well taught.

With this in mind, the various mathematical organizations in the United States cooperated in the formation of the School Mathematics Study Group (SMSG). SMSG includes college and university mathematicians, teachers of mathematics at all levels, experts in education, and representatives of science and technology. The general objective of SMSG is the improvement of the teaching of mathematics in the schools of this country. The National Science Foundation has provided substantial funds for the support of this endeavor.

One of the prerequisites for the improvement of the teaching of mathematics in our schools is an improved curriculum--one which takes account of the increasing use of mathematics in science and technology and in other areas of knowledge and at the same time one which reflects recent advances in mathematics itself. One of the first projects undertaken by SMSG was to enlist a group of outstanding mathematicians and mathematics teachers to prepare a series of textbooks which would illustrate such an improved curriculum.

The professional mathematicians in SMSG believe that the mathematics presented in this text is valuable for all well-educated citizens in our society to know and that it is important for the precollege student to learn in preparation for advanced work in the field. At the same time, teachers in SMSG believe that it is presented in such a form that it can be readily grasped by students.

In most instances the material will have a familiar note, but the presentation and the point of view will be different. Some material will be entirely new to the traditional curriculum. This is as it should be, for mathematics is a living and an ever-growing subject, and not a dead and frozen product of antiquity. This healthy fusion of the old and the new should lead students to a better understanding of the basic concepts and structure of mathematics and provide a firmer foundation for understanding and use of mathematics in a scientific society.

It is not intended that this book be regarded as the only definitive way of presenting good mathematics to students at this level. Instead, it should be thought of as a sample of the kind of improved curriculum that we need and as a source of suggestions for the authors of commercial textbooks. It is sincerely hoped that these texts will lead the way toward inspiring a more meaningful teaching of Mathematics, the Queen and Servant of the Sciences.

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PREFACE

The aim of this experimental text is to focus attention on mathematical ideas which are appropriate for study by college-capable students in the eleventh grade. These ideas have been selected and developed by mathematicians and teachers working together. The mathematicians believe that the mathematics presented is significant, both intrinsically and as preparation for future study, and that the expositions are mathematically sound. The teachers believe that the material is teachable to high school students. Both groups join in the belief that there is an affinity between youth and clearly presented mathematics which should be more thoroughly exploited in the nation's schools. The success of this text will depend, in large measure, on the extent to which it stimulates students' interest and influences them to continue their study of mathematics in high school and subsequently in college.

In this text students encounter many new mathematical ideas which require expositions somewhat more sophisticated than those previously attempted. These expositions develop the idea that mathematics is an organized body of knowledge which is founded on a surprisingly small number of basic assumptions. Students who become aware of this important idea will begin to understand the structure of mathematics and will acquire some ability to explore this structure for themselves.

Explanations which emphasize proof require intensive study. For this reason no claim is made that this is a course in "mathematics made easy". On the contrary, inherent difficulties are candidly appraised and forthrightly explained in terms that are appropriate for students at this grade level. For this purpose the easiest or shortest presentation is not always the best. For example, the rules for solving systems of equations could have been given in much less space than is devoted to the development of equivalent systems in Chapter Seven; but this development provides a logical basis for understanding these rules. Again, the rules for the manipulation of complex numbers could have been stated briefly rather than derived from carefully chosen postulates as they are in Chapter Five. Similar examples can be found in every chapter, indeed, in almost every section of this text. The purpose in all such cases is to give the student some insight into the nature of mathematical thought as well as to prepare him to perform certain manipulations with facility.

The course of study in grade eleven was greatly improved as a result of the Text Book Panel's decision to devote only one year (grade ten) to plane and solid geometry. The time gained by the removal of solid geometry from the eleventh grade sequence is devoted to trigonometry (Chapter X), vectors, (XI), and a more extensive treatment of complex numbers (V, IXX) than is ordinarily attempted at this stage. The sequence of topics in this sample text is, of course, only one of many that could have been chosen. One controlling consideration here was the desire to advance the student's understanding of number systems. While this development permeates the entire text, its main bearings are to be found in Chapter I (Number Systems), Chapters V and XII (Complex Numbers), and, for the very able student, Chapter XV (Algebraic Structures).

The writing group hopes that the following viewpoints are discernible in this text.

Plausible arguments have their place provided they do not implant ideas which must later be eradicated. The necessity for improving the student's understanding of the nature of mathematical reasoning does not imply that every argument must take the form of a rigorous proof.

It is often desirable to appeal to the student's intuition and to lead him by an inductive approach to make and test conjectures about the nature of the principles to be proved.

New symbolism should never be used for the sake of being "modern" but only when it serves to convey meaning more accurately and more succinctly than could be done by other means.

Individual differences in ability and motivation must be recognized even among college-capable students. Some material must be included for the student who has exceptional ability in mathematics.

This revision of the original (1959) version of this text was based upon a careful study of the suggestions and evaluations which were submitted by the teachers who taught this material in the experimental centers during the 1959-60 academic year. In a very real sense these teachers collaborated with the authors in an effort to make this text a more effective instrument of instruction.

Chapter 1

NUMBER SYSTEMS

1-1. Introduction.

This chapter is about the number systems of elementary algebra. You are already familiar with some of these number systems. You have used the natural numbers, 1,2,3 ... , ever since you started to count. The set of integers, ... -3,-2,-1,0,1,2,3, ..., contains all the natural numbers and has zero and the negative integers as well. You probably met this number system for the first time, in a serious way, when you began to study algebra. The system of rational numbers is an even richer system. It contains all those numbers of the form $\frac{p}{q}$, where p and q are integers and q is not zero. You were studying the positive rational numbers when you learned to work with fractions. However you did not meet the negative numbers until you began your study of algebra.

In this chapter we shall meet still another number system called the real number system. Before we take up our study of the real number system we are going to examine again the natural numbers, the integers and the rational numbers. In this reexamination we shall study the logical "structure" of the various systems of numbers. We show how the study of this structure ties together all of the many special facts about the different systems which together make up the real number system. This program is carried further in Chapter 5 where complex numbers are studied.

When we speak of the logical structure of a number system we have in mind a very definite meaning which requires explanation. About 300 B.C. Euclid organized geometry as a logical system, selecting certain statements as axioms or postulates and deducing from them other statements called theorems. It was relatively recently--within the last 100 years or so--that a similar organization of algebra and arithmetic was undertaken.

Organizing number systems in deductive form gives our knowledge of them a coherence we might not otherwise find. We shall see that each system can be summarized "in a nutshell" by listing its "basic properties"--from which we may derive other properties--and that these "summaries" make it easy to see the pattern common to all of them and to see their differences.

The important idea in the logical structure of a mathematical system is that some statements are consequences of other statements. Many of the theorems reflect this idea themselves since they state that one statement follows from another. They have the form

(1-1a) If A, then B

where the letters A and B stand for certain statements. We shall meet many such theorems in this chapter and elsewhere in this book. Therefore we consider them in some detail in this introductory section. Examples of theorems having this form may be given by taking specific statements for A and B.

Thus with

A: triangles T_1 and T_2 are congruent

B: triangles T_1 and T_2 are similar

we have the theorem

If triangles T_1 and T_2 are congruent, then triangles T_1 and T_2 are similar.

And with

A: x is negative

B: x^2 is positive

we have the theorem

If x is negative, then x^2 is positive.

When a theorem has the form "If A, then B" the problem of "proving" it demands that a chain of reasons be given to convince the reader that statement B is true if statement A is true. Note that no assertion is made as to whether either A or B actually is true; only that in case one of them (A) happens to be true, then the other (B) must also be true. In case A is not true, (1-1a) has nothing whatever to say about B .

If both a theorem "If A, then B" and its converse "If B, then A" are true, this fact is often expressed by saying

A if and only if B

which is interpreted as meaning

A if B, and A only if B.

In the latter manner of expression, "A if B" stands for "If B, then A", while "A only if B" stands for "If not B then not A", or equivalently, "If A then B".

This is the way we shall use the expression "only if" in this book. It is important for you to remember this fact. For example the "only if" statement

$$x = y \text{ only if } x^2 = y^2$$

has for us the same meaning as the "if then" statement

$$\text{"If } x = y \text{ then } x^2 = y^2\text{"}$$

which happens to be true for all x and y . Our "only if" statement does not have the same meaning as

$$\text{"If } x^2 = y^2 \text{ then } x = y\text{"}$$

which is not true for all x and y .

Exercises 1-1a

1. Form the converse of each of the following statements:
   (a) If $x + 3 = y$, then y is greater than x .
   (b) A natural number is a multiple of 2 if it is even.

(c) $x = 1$ only if $x^2 = 1$.
(d) If x is less than y, then x is less than z .
(e) "If A, then B" is the converse of "If B, then A" only if "If B, then A" is the converse of "If A, then B" .

2. Rephrase each of the following in the form "If A, then B; and if B, then A ."
   (a) $x = y$ if and only if $x + z = y + z$
   (b) $x + 1 = y$ if and only if $y - 1 = x$
   (c) $2x + 1 = 7$ if and only if $x = 3$
   (d) $(x + y)^2 = x^2 + y^2$ if and only if x or y is zero
   (e) The converse of "If A, then B" is true if and only if "If B, then A" is true.

1-2. The System of Natural Numbers.

The elements of the natural number system are the numbers 1,2,3 ... , the numbers used in counting. The numbers of this system are ordered in a familiar way; the first is 1 , the second, 2 , is obtained by adding 1 to 1 , the third, 3 , is obtained by adding 1 to 2 , and so on. We use one letter N to denote the natural number system.

If a and b are any natural numbers then we can add these numbers to obtain their sum $a + b$ and we can multiply these numbers to obtain their product ab . For some pairs of natural numbers a and b we can also subtract to obtain the natural numbers $a - b$ but not for all pairs. For instance 5 and 3 are natural numbers and so is $5 - 3$, but not $3 - 5$. For some pairs of natural numbers a and b we can also divide to obtain a natural number $\frac{a}{b}$ but not for every pair. For instance 6 and 3 are natural numbers and so is $\frac{6}{3}$ but not $\frac{3}{6}$.

The operations of addition and multiplication can be performed with any two natural numbers to yield natural numbers. The operations of subtraction and division do not have this property. We express these facts by saying that the natural number system is closed under addition and multiplication but not under subtraction or division.

This property of a number system, of being "closed" with respect to an operation, is one we shall meet over and over again. As we proceed we will find that each new system we encounter is closed under more operations than any of its predecessors.

Exercises 1-2a

Here are some sets of natural numbers. For each decide whether it is closed under addition, multiplication, subtraction, division.

1. The set of all natural numbers .
2. The set of all even natural numbers.
3. The set of all odd natural numbers.
4. $\{1,2,3,4,5\}$
5. $\{0,1\}$
6. The set of all natural numbers greater than 17 .

Any given natural number may be described in a variety of ways. Thus 4, $1 + 3$, $2 + 2$, $2 \cdot 2$, $1 \cdot 4$ are all descriptions of the same number. We call this relation equality and express it using the sign "=". Thus we write $4 = 1 + 3$, $4 = 2 + 2$, $2 + 2 = 2 \cdot 2$. Given any pair of symbols a,b representing numbers, there are only two possibilities: either they are "equal" $(a = b)$, or they are not. In the latter case we say that a and b are different or distinct and we write $a \neq b$.

The general rules governing the use of the equality sign are

$\underline{E}_1$ (Dichotomy). Either $a = b$ or $a \neq b$.

$\underline{E}_2$ (Reflexivity). $a = a$.

$\underline{E}_3$ (Symmetry). If $a = b$, then $b = a$

$\underline{E}_4$ (Transitivity). If $a = b$ and $b = c$, then $a = c$.

$\underline{E}_5$ (Addition). If $a = b$, then $a + c = b + c$.

$\underline{E}_6$ (Multiplication). If $a = b$, then $ac = bc$.

These rules give directions for asserting certain statements of equality, in each case but the first two, when certain other statements of equality are either given or supposed. In $\underline{E}_1, \dots , \underline{E}_6$ each of the letters a,b,c is to be understood as representing any one of the numbers in the system N . The point here is that no matter what numbers a, b, and c represent, if (for example) $a = b$ and $b = c$, then it follows that $a = c$. We express the important fact that there is complete freedom in substituting for the letters a,b,c by saying they are arbitrary.

The operations of addition and multiplication in the natural number system have the following properties (among others).

$\underline{A}_1$ (Closure). $a + b$ is a natural number.

$\underline{A}_2$ (Commutativity). $a + b = b + a$.

$\underline{A}_3$ (Associativity). $a + (b + c) = (a + b) + c$.

$\underline{M}_1$ (Closure). ab is a natural number.

$\underline{M}_2$ (Commutativity). $ab = ba$.

$\underline{M}_3$ (Associativity). $a(bc) = (ab)c$.

$\underline{M}_4$ (Multiplicative Identity). $1 \cdot a = a \cdot 1 = a$.

D (Distributivity). $a(b + c) = ab + ac$.

C_1 (Cancellation-Addition). If $a + c = b + c$, then $a = b$.

C_2 (Cancellation-Multiplication). If $ac = bc$, then $a = b$.

These properties, which we shall call the E,A,M,D,C properties, are general statements of familiar "laws" of arithmetic; they are "general" in the sense that we assert their validity for arbitrary a, b, c . Some of the corresponding "special" statements in arithmetic are $2 + 3 = 3 + 2$, $7(5 + 1) = 7 \cdot 5 + 7 \cdot 1$.

Exercises 1-2b

1. Which one of the natural number properties is illustrated by each of the following statements? (All letters represent arbitrary natural numbers.)
   - (a) $4 + 5 = 5 + 4$
   - (b) $8(x + 2) = 8x + 16$
   - (c) $3(4 \cdot 7) = (3 \cdot 4) \cdot 7$
   - (d) $xy + xz = x(y + z)$
   - (e) $7 \cdot 45 = 280 + 35$
   - (f) $(x + 2) + 3 = x + 5$
2. Using the natural number properties, prove the following statements to be true for all natural numbers.
   - (a) $(x + y)z = xz + yz$
   - (b) $x + xy = x(1 + y)$
   - (c) $x[y + (w + z)] = x(y + w) + xz$
   - (d) If $x + (y + z) = (z + y) + xz$, then $x = xz$

*3. Use properties E,A,M to prove the statements:
If $a = b$ and $c = d$, then $a + c = b + d$.
If $a = b$ and $c = d$, then $ac = bd$.

We examine a few consequences of properties E,A,M,D,C.

First of all, property $\underline{A}_3$ (Associativity) asserts that for arbitrary a, b, c we have $a + (b + c) = (a + b) + c$. Consider on the other hand the expression $a + b + c$. We ordinarily use the sign "+" to denote an operation which assigns one natural number (their sum) to each pair of given natural numbers, and therefore we should hesitate to use it when more than two numbers are involved. However, such hesitation is unnecessary, since the associative property tells us that it makes no difference at all whether parentheses are inserted around the first two terms or around the last two. It is thus precisely because of the associative law that we may define the expression $a + b + c$ to be a third description of the one number already having the two names $a + (b + c)$ and $(a + b) + c$:

$$a + b + c = (a + b) + c \ .$$

Similar definitions can be made for expressions with more terms, such as $a + b + c + d$.

We may adapt the distributive property to sums involving more than two terms:

$$\begin{aligned} a(b + c + d) &= a((b + c) + d) && \text{[Definition} \\ &= a(b + c) + ad && \text{[Distributivity} \\ &= (ab + ac) + ad && \text{[Distributivity} \\ &= ab + ac + ad \ . && \text{[Definition} \end{aligned}$$

Now consider the expressions $a + a$ and $a + a + a$. By property $\underline{M}_4$ (Multiplicative Identity) each term in these expressions equals $a \cdot 1$. Using the extended distributive properties, we can say

$$a + a = a \cdot 1 + a \cdot 1 = a(1 + 1) = 2a \ ,$$

$$a + a + a = a(1 + 1 + 1) = 3a \ .$$

In general,

$$na = a + a + \dots + a ,$$

where there are n terms, n being any natural number.

Similar considerations apply to products, so that we may define

$$abc = (ab)c ,$$
$$abcd = (abc)d ,$$

and similarly with more factors.

Corresponding to the expressions $2a$, $3a$ for sums we abbreviate products of like factors as

$$a \cdot a = a^2$$
$$a \cdot a \cdot a = a^3 .$$

In general

$$a^n = a \cdot a \cdot \dots a ,$$

when there are n factors, n being any natural number.

Example 1-2a: Using properties E,A,M,D prove that for arbitrary a , b in N ,

$$(a + b)^2 = a^2 + 2ab + b^2 .$$

Proof:
$$\begin{aligned} (a + b)^2 &= (a + b)(a + b) && \text{[Def.} \\ &= (a + b)a + (a + b)b && \text{[Distr.} \\ &= (aa + ba) + (ab + bb) && \text{[Distr.} \\ &= a^2 + ab + ab + b^2 && \text{[Def., Comm.} \\ &= a^2 + 2ab + b^2 && \text{[Def.} \end{aligned}$$

Exercises 1-2c

1. Using the natural number properties, remove all parentheses from products and list the properties you use.

   (a) $5p(3 + r)$
   (b) $(2x + 3)(x + 4)$
   (c) $(y + 1)(y + 1)$
   (d) $2m(m + n + 3)$
   (e) $(x + 1)(x + y + 2)$

2. Prove that the following statements are true where all letters represent arbitrary natural numbers.

   (a) $(a + b + c) + d = (a + b) + (c + d)$
   (b) $(a + b)(c + d) = ac + ad + bc + bd$
   (c) $(px + q)(rx + t) = prx^2 + (pt + qr)x + qt$
   (d) $a(b + c + d) = ab + ac + ad$
   (e) $a(bcd) = (ab)(cd)$

3. Using natural number properties, simplify the following to a single term.

   (a) $4x + 2xy$
   (b) $2(4u + 1) + 3(4u + 1)$
   (c) $m(p + q) + m(p + q)$
   (d) $(2x + 1)(x + 1) + (1 + 2x)(1 + x)$

4. Prove that the square of an even natural number is also an even natural number.

5. Prove that the square of an odd natural number is also an odd natural number.

6. Is the product of an even natural number and an odd natural number even or odd? Prove your answer.

7. Since $15^2 = 225$, $25^2 = 625$, $35^2 = 1225$, ... , $85^2 = 7225$, $95^2 = 9025$, a pattern can be seen that a two digit natural number ending in 5 can be squared by writing the product of the first digit by one more than the first digit, and following this the square of 5 . Prove that this is true without testing every case.

The lists $\underline{E}$,$\underline{A}$,$\underline{M}$,$\underline{D}$,$\underline{C}$ of properties of N , when taken together with another list, $\underline{O}$, to be presented in Section 1-3, form a logical basis of the natural number system. In organizing the natural number system deductively these basic properties may be assigned the role played by the axioms and postulates in the deductive organization of geometry. From them we may derive as theorems the other algebraic properties of the natural number system. Corresponding lists of basic properties for the systems of the integers, the rationals, and the reals are in later sections of this chapter; and for the complex number system, in Chapter 5.

Limitations of space prevent us from going very far into this "deductive theory" of number systems, but a few examples will be given to illustrate the methods by which some of the familiar "rules of calculation" may be derived from the $\underline{E}$,$\underline{A}$,$\underline{M}$,$\underline{D}$,$\underline{C}$ list. Beginning in the next section we shall study inequalities from the deductive point of view.

Example 1-2b: Solve the equation $5x + 3 = 13$, and justify each step in the solution using properties $\underline{E}$,$\underline{A}$,$\underline{M}$,$\underline{D}$,$\underline{C}$ of the natural number system.

Proof: From the (arithmetical) fact that $13 = 10 + 3$, we use $\underline{E}_4$ to rewrite $5x + 3 = 13$ as

$$5x + 3 = 10 + 3 .$$

Then by $\underline{C}_1$ (Cancellation-Addition)

$$5x = 10 .$$

Again, an arithmetical fact: $10 = 5 \cdot 2$; and again we rewrite:

$$5x = 5 \cdot 2 .$$

Finally, using $\underline{C}_2$ (Cancellation-Multiplication), we get

$$x = 2 .$$

Check: Substitution of 2 for x in the original equation shows that it is satisfied:

$$5 \cdot 2 + 3 = 10 + 3 = 13 .$$

Comment on Example 1-2b: When we study other number systems (the integers, the rationals, etc.) we shall be able to attack this problem more directly than the means presently available permit--for the reason that we shall have longer lists of basic properties to work with. The significant thing about Example 2 is that it can be solved at all in the system N . The "method" we used is rather involved--the only virtue we claim for it is that it can actually be carried out using only the E,A,M,D,C properties of N . In the system I of integers, where we have the number -3 available we shall be able to add -3 to both sides of $5x + 3 = 13$ to get $5x = 10$ directly. And in the rational number system Q where we have the number $\frac{1}{5}$ available, we can then multiply both sides of $5x = 10$ by $\frac{1}{5}$ to get $x = 2$. We can perform neither of these steps in N since N contains neither -3 nor $\frac{1}{5}$.

Exercises 1-2d

Find natural number solutions for the following, and name the natural number properties E,A,M,D,C used.

1. $x + 2 = 5$
2. $z + 3 = 1$
3. $3y = 6$
4. $2u + 5 = 7$
5. $2u + 1 = 4$
6. $3p + 4 = 4 + p$
7. $2w + 1 = 4 + 3w$
8. $3m + 1 = 2m + 4$

1-3. Order in the Natural Number System.

One of the first things--if not *the* first thing--one learns about the natural numbers is that they come in a definite order: 1, then 2, then 3, then 4, etc. This is the order of counting. When a natural number a "precedes" a natural number b in the order of counting, we say "a is less than b" and write $a < b$. When $a < b$ we also say "b is greater than a" and write $b > a$. Thus $a < b$ and $b > a$ have exactly the same meaning. Moreover each of these statements has the same meaning as the statement: there is a natural number c such that $a + c = b$. Thus there is a very close connection between the order relation of natural numbers and the operation of addition.

We examine some basic properties of this order relation. The first property expresses the fact that given any pair of distinct natural numbers, one or the other of them is the greater:

$\underline{O}_1$ (Trichotomy). Given any pair a, b of natural numbers, exactly one of the following three relations holds:

$$a = b, \qquad a < b, \qquad b < a.$$

$\underline{O}_2$ (Transitivity). If $a < b$ and $b < c$, then $a < c$.

$\underline{O}_3$ (Addition). If $a < b$, then $a + c < b + c$.

$\underline{O}_4(N)$ (Multiplication). If $a < b$, then $ac < bc$. (c in N)

The first property is called the "trichotomy property" because it splits the possibilities into three parts, one of which must hold while no two of which may both hold. Note, in particular, that if $b < a$ is false, then either $a = b$ or $a < b$, which is written concisely as $a \leq b$.

Similarly $a \geq b$ is written in place of "$a = b$ or $a > b$". Note also that "$a < b$ or $b < a$" means simply "$a \neq b$". If $a < b$ and $b < c$ we often write $a < b < c$, in analogy to $a = b = c$ for $a = b$ and $b = c$. Writing such "chains" of equalities and inequalities is justified by the transitivity properties $\underline{E}_4$ and $\underline{O}_2$. Thus in the chain of equalities

$$a = b = c = d = \ldots$$

each member equals each of the others. In a chain of inequalities

$$a < b < c < d \ldots$$

each number is less than each of those following it.

Using the connection between order and addition,

$$a < b \quad \text{if and only if} \quad \begin{cases} \text{there is a } c \text{ in } N \\ \text{such that } a + c = b, \end{cases}$$

$\underline{O}_2$ says

If there are d and e in N for which

$$a + d = b \quad \text{and} \quad b + e = c,$$

then there is an f in N such that $a + f = c$.

This restatement of $\underline{O}_2$ is easily proved by showing that $d + e$ is such an f: if

$$a + d = b \quad \text{and} \quad b + e = c,$$

then

$$(a + d) + e = b + e = c,$$

so

$$a + (d + e) = c, \text{ where } d + e \text{ is in } N.$$

In a similar way, $\underline{O}_3$ and $\underline{O}_4(N)$ may be restated in terms of addition and proved from the $\underline{E}$,$\underline{A}$,$\underline{M}$,$\underline{D}$ properties of N.

The designation $\underline{O}_4(N)$ is used rather than simply $\underline{O}_4$ to warn the reader that this particular property will require modification in the other number systems to be studied in this chapter.

Exercises 1-3a

1. List the members of the set of natural numbers such that $x < 5$.
2. Using natural numbers, write an equality having the same meaning as $6 > 2$.
3. Using the symbol "<", form true statements using the following pairs of natural numbers.
   - (a) 2 and 6
   - (b) 5 and 3
   - (c) a and $3a$
   - (d) $(2 + a)$ and $(1 + a)$
   - (e) c and b , if $c = a + b$
   - (f) a and e , when $a + b = c$ and $c + d = e$.
4. Rewrite the following statements using $a < b < c$, $a \leq b$, or $a \neq b$ forms:
   - (a) x is less than 4 or x is equal to 4 .
   - (b) 5 is less than x and x is less than 7 .
   - (c) y is equal to 4 or y is greater than 4 .
   - (d) m is less than n or n is less than m .
   - (e) 3 is less than x or 3 is equal to x , and x is less than 5 or x is equal to 5 .
5. Restate $\underline{O}_3$ and $\underline{O}_4$ in terms of addition and prove them from the $\underline{E},\underline{A},\underline{M},\underline{D}$ properties of N .
6. If $x + a = y$ and $y + b = z$ (all letters representing arbitrary natural numbers), what is the order relation of x and z ?

There are some similarities between the $\underline{E}$ properties and the $\underline{O}$ properties which deserve to be noted, as they reveal analogies between the methods of treating equations and corresponding methods for inequalities.

The two equality properties

$\underline{E}_5$: If $a = b$, then $a + c = b + c$

$\underline{E}_6$: If $a = b$, then $ac = bc$

correspond exactly to the order properties

$\underline{O}_3$: If $a < b$, then $a + c < b + c$

$\underline{O}_4(N)$: If $a < b$, then $ac < bc$ (c in N)

and it is on the basis of this correspondence that a theory of inequalities may be built to parallel that for equations.

Fully as important in practice as $\underline{E}_5$, $\underline{E}_6$, $\underline{O}_3$, $\underline{O}_4(N)$ are their converses, the cancellation properties for equality and order:

$\underline{C}_1$: If $a + c = b + c$, then $a = b$

$\underline{C}_2$: If $ac = bc$, then $a = b$ (c in N)

$\underline{C}_3$: If $a + c < b + c$, then $a < b$

$\underline{C}_4$: If $ac < bc$, then $a < b$ (c in N)

Of these, the first two were included in the lists of basic properties for N . All four, however, may be proved as theorems using the $\underline{E}$,$\underline{A}$,$\underline{M}$ and $\underline{O}$ properties of N . We examine one of these proofs to show the power of the trichotomy property.

Theorem 1-3a: ($\underline{C}_1$) If $a + c = b + c$, then $a = b$.

Proof: We suppose that $a + c = b + c$ and deduce $a = b$ from this assumption. By $\underline{O}_1$ (Trichotomy), there are exactly three possibilities, one of which must hold; they are

$a = b$, $a < b$, $b < a$.

If we can eliminate the last two possibilities, the first must hold and the theorem is true. We therefore suppose $a < b$. By $\underline{O}_3$ (Addition) it follows that $a + c < b + c$ which (by $\underline{O}_1$ again!) flatly contradicts our hypothesis $a + c = b + c$. Similarly supposing $b < a$ also leads to a contradiction. Thus if $a + c = b + c$, it follows that $a = b$.

Theorem 1-3b: ($\underline{C}_2$) If $ac = bc$, then $a = b$ (c in N).

Theorem 1-3c: ($\underline{C}_3$) If $a + c < b + c$, then $a < b$.

Theorem 1-3d: ($\underline{C}_4$) If $ac < bc$, then $a < b$ (c in N).

The proofs are similar to that for Theorem 1-3a and are left as exercises.

Before we use these theorems to solve inequalities, we note that because they are the converses of $\underline{E}_5$, $\underline{E}_6$, $\underline{O}_3$, $\underline{O}_4$(N) we may express all eight of these properties in the four compound statements:

$\underline{EC}_1$: $a = b$ if and only if $a + c = b + c$

$\underline{EC}_2$: $a = b$ if and only if $ac = bc$ (c in N)

$\underline{OC}_1$: $a < b$ if and only if $a + c < b + c$

$\underline{OC}_2$: $a < b$ if and only if $ac < bc$ (c in N).

If, in solving an equation or inequality, we use only the $\underline{C}$ properties (as we did in Example b, Section 1-2), our discussion is not logically complete until we perform the "check." For until we do this, we do not know whether there *is* any solution. The compound forms $\underline{EC}$, $\underline{OC}$ are important in practice because they guarantee that each step in which they are used is reversible.

Thus $\underline{C}_1$ gives

$$\text{If } x + 3 = 5 \text{ , then } x = 2 \text{ ,}$$

which *does* *not* say that $x + 3 = 5$ has any solution; only that *if* it has a solution, then that solution must be 2 . On the other hand $\underline{E}_5$ says

$$\text{If } x = 2 \text{ , then } x + 3 = 5 \text{ ,}$$

which asserts that 2 *does* indeed satisfy $x + 3 = 5$. (This is the "check.") The compound statement $\underline{EC}_1$ gives

$$x + 3 = 5 \quad \text{if and only if} \quad x = 2 \text{ ,}$$

including *both* assertions: (i) $x + 3 = 5$ is satisfied by 2 ("if" part), (ii) no other number satisfies $x + 3 = 5$ ("only if" part).

We illustrate these theorems by solving an inequality.

Example 1-3a: Solve $5x + 3 < 13$ in the system N .

Solution: (The method is much the same as that used in solving Example 1-2b, except that we now use the order properties corresponding to the equality properties used there.)

$5x + 3 < 10 + 3$. if and only if $5x < 10$ [$\underline{OC}_1$

$5x < 5 \cdot 2$ if and only if $x < 2$ [$\underline{OC}_2$

$x < 2$ if and only if $x = 1$

There are two ways to attack an inequality like $5x + 3 \leq 13$. One of them is to split it into two problems:

$$5x + 3 < 13 \qquad \text{or} \qquad 5x + 3 = 13$$

and to solve them separately:

$$x < 2 \qquad \text{or} \qquad x = 2 \text{ .}$$

This split can be avoided if we combine $\underline{EC_1}$, $\underline{OC_1}$ and $\underline{EC_2}$, $\underline{OC_2}$ obtaining

$\underline{EOC_1}$: $a \leq b$ if and only if $a + c \leq b + c$

$\underline{EOC_2}$: $a \leq b$ if and only if $ac \leq bc$ (c in N)

With these last compound statements, inequalities involving $\leq$ may be handled just like the others. Thus, for example,

$5x + 3 \leq 13$ if and only if $5x \leq 10$ [$\underline{EOC_1}$

if and only if $x \leq 2$. [$\underline{EOC_2}$

Exercises 1-3b

1. Prove the following properties of N , where a, b, c and d represent arbitrary natural numbers.
   (a) If $a + b = c$, then $a + b < c + b$
   (b) If $a(b + c) = d$, then $ab < d$
   (c) If $a < b$ and $c < d$, then $a + c < b + d$
   (d) If $ac = bc$, then $a = b$ (Theorem 1-3b)
   (e) If $a + c < b + c$, then $a < b$ (Theorem 1-3c)
   (f) If $ac < bc$, then $a < b$ (Theorem 1-3d)
2. Solve the following for natural numbers.
   (a) $2m < 4$
   (b) $6p + 3 < 15$
   (c) $3x + 1 \leq 4$
   (d) $3x + 4 < x + 8$
   (e) $5y + 17 \geq 9y + 1$
   (f) $4 < 3x + 1 < 19$
3. Prove: $a < b < c$ if and only if $a + d < b + d < c + d$.

Two more order properties are required for a logical basis for the natural number system. They are the so-called "Archimedian property" and the "well order property." They are stated below.

These properties are basic for much of the advanced theory of the natural numbers, some of which is beyond the scope of this book. The well order property is a property of the order relation in the natural number system which does not hold for any of the other number systems discussed in this chapter. On the other hand the Archimedian property holds for all of the number systems considered in this chapter--provided the second occurrence of the word "natural" is replaced by "positive."

LIST OF BASIC PROPERTIES
OF THE
NATURAL NUMBER SYSTEM

In the following general statements, a, b, c represent arbitrary members of N :

$\underline{E}_1$ (Dichotomy) Either $a = b$, or $a \neq b$.

$\underline{E}_2$ (Reflexivity) $a = a$.

$\underline{E}_3$ (Symmetry) If $a = b$, then $b = a$.

$\underline{E}_4$ (Transitivity) If $a = b$ and $b = c$, then $a = c$.

$\underline{E}_5$ (Addition) If $a = b$, then $a + c = b + c$.

$\underline{E}_6$ (Multiplication) If $a = b$, then $ac = bc$.

$\underline{A}_1$ (Closure) $a + b$ is a natural number .

$\underline{A}_2$ (Commutativity) $a + b = b + a$.

$\underline{A}_3$ (Associativity) $a + (b + c) = (a + b) + c$.

$\underline{M}_1$ (Closure) ab is a natural number.

$\underline{M}_2$ (Commutativity) $ab = ba$.

$\underline{M}_3$ (Associativity) $a(bc) = (ab)c$.

$\underline{M}_4$ (Multiplicative Identity) $1 \cdot a = a \cdot 1 = a$.

$\underline{D}$ (Distributivity) $a(b + c) = ab + ac$.

$\underline{O}_1$ (Trichotomy) Exactly one of the following holds:
$a = b$, $a < b$, $b < a$.

$\underline{O}_2$ (Transitivity) If $a < b$ and $b < c$, then $a < c$.

$\underline{O}_3$ (Addition) If $a < b$, then $a + c < b + c$.

$\underline{O}_4$ (N) (Multiplication) If $a < b$, then $ac < bc$.

$\underline{O}_5$ (Archimedes) If a and b are any given natural numbers such that $a < b$, there is a natural number n such that $na > b$.

$\underline{O}_6$ (Well Order) Each set of one or more natural numbers contains a minimal member; i.e., a member less than or equal to every member of the set.

DEFINITIONS FOR THE NATURAL NUMBER SYSTEM

In the following general statements, a, b, c, d represent arbitrary members of N :

$a = b$ if and only if a and b are names for the same number.

$a + b + c = (a + b) + c$, $a + b + c + d = (a + b + c) + d$, and similarly with more terms.

$na = a + a + \ldots + a$, where there are n terms, n in N .

$abc = (ab)c$, $abcd = (abc)d$, and similarly for more factors.

$a^n = a \cdot a \cdot \ldots \cdot a$, where there are n factors, n in N .

$a < b$ if and only if there is an e in N such that $a + e = b$.

$b > a$ if and only if $a < b$.

$a \leq b$ if and only if $a < b$ or $a = b$.

$a < b < c$ if and only if $a < b$ and $b < c$.

SOME THEOREMS OF THE NATURAL NUMBER SYSTEM

In the following general statements, a, b, c, d represent arbitrary members of N :

$a = b$ if and only if $a + c = b + c$ ($\underline{EC_1}$)

$a = b$ if and only if $ac = bc$ ($\underline{EC_2}$) (c in N)

$a < b$ if and only if $a + c < b + c$ ($\underline{OC_1}$)

$a < b$ if and only if $ac < bc$ ($\underline{OC_2}$) (c in N)

$a < b < c$ if and only if $a + d < b + d < c + d$.

Exercises 1-3c

1. Use the natural numbers 1, 2, and 3 to illustrate $\underline{O}_5$ (Archimedes).
2. Which element of the set of two digit natural numbers is the minimal element guaranteed by $\underline{O}_6$ (Well-order)?
3. Which of the $\underline{E},\underline{A},\underline{M},\underline{D},\underline{O}$ properties of the natural numbers are best illustrated by the following statements? All letters represent arbitrary natural numbers.
   (a) $(x + y)(2x + 3y) = (2x + 3y)(x + y)$
   (b) $2 + 4 = 3 \times 2$
   (c) $(a + b) + (c + 2) = (a + b + c) + 2$
   (d) $4y(y + 1) = 4y^2 + 4y$
   (e) $2 < a$ and $a < b$, so $2 < b$
   (f) $2[5(x + y)] = 10(x + y)$
   (g) $2(m + n) < T(m + n)$ if $2 < T$
   (h) $(a + b)^2 = a^2 + b^2$ or $(a + b)^2 \neq a^2 + b^2$
   (i) $a^2 + b^2 + 2ab = a^2 + 2ab + b^2$
   (j) $x + 2 = y + 2$ if $x = y$
   (k) $p = q$ only if $p + m = q + m$
   (l) $5 \leq 2x + y$ or $5 > 2x + y$
   (m) If $u = v$, then $u^2 = v^2$
   (n) $2y < qy$ if $2 < q$
   (o) If $(x + y) < 4$ then $z(x + y) > 4$ is true for some z.
4. Prove that $x + 2 = 2$ cannot be solved in the natural number system. (Hint: assume a natural number, say p, is a solution and apply the definition for $a < b$).

1-4. The System of Integers.

The system I of integers has as its members the numbers

$$\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots .$$

It includes as a part the system of natural numbers as well as the number 0 and the "negative" whole numbers. (Why these numbers are called "negative" will appear in Section 1-5 when we study the order relation for I .)

In I we can solve equations such as

$$2 + x = 2 \quad \text{and} \quad 2 + x = 1$$

which cannot be solved in the system N of natural numbers. However, we can do more than this in I ; we can solve any equation of the form $a + x = b$ where a and b are any members of I , whether or not they are in N as well.

The system I has all of the E,A,M,D properties of N and, in addition, two more A properties:

A_4 (Additive Identity) $a + 0 = a$, for arbitrary a in I

A_5 (Subtraction) For each pair a, b of integers in I , there is exactly one integer c such that $a + c = b$.

Definition 1-4a: $c = b - a$ means $a + c = b$, and $b - a$ is called the difference of b and a (in that order).

The process of solving the equation $a + x = b$ may be interpreted as performing a new operation, subtraction. With this interpretation, A_5 asserts that I is closed under subtraction.

An important special case of A_5 and Definition 1-4a is that in which $b = 0$. In this case the solution of the equation $a + x = 0$ is given a special name.

Definition 1-4b: The solution of $a + x = 0$ is denoted by $-a$ and is called the additive inverse of a.

Thus if a is any member of I, we have $a + (-a) = 0$. Moreover in view of Definition 1-4a we have $-a = 0 - a$.

Theorem 1-4a: $-(-a) = a$, for arbitrary a in I.

Proof: By $\underline{A}_5$ there is exactly one number in I satisfying $(-a) + x = 0$ and by Definition 1-4b that number is $-(-a)$. However $a + (-a) = 0$ and hence using $\underline{A}_2$ (Commutativity), $(-a) + a = 0$. But $(-a) + a = 0$ is simply the assertion that a itself is a solution of $(-a) + x = 0$. Thus both a and $-(-a)$ satisfy the equation $(-a) + x = 0$. And since there can be only one solution we conclude $a = -(-a)$.

> The crux of this proof (and of most of the others in this section) is that whenever two expressions satisfy an equation which has only one solution they must be equal.

Those non-zero integers which are not natural numbers (-1, -2, -3, ...) are the additive inverses of the natural numbers. Hence we have the following corollary to Theorem 1-4a.

Corollary 1-4a: If a is a non-zero integer, either a is a natural number, or $-a$ is a natural number.

Exercises 1-4a

1. Find additive inverses for the following integers:

(a) 2	(d) m
(b) -5	(e) $-p$
(c) 0	(f) $(b - a)$

2. Which of the properties, definitions, or theorems for I are illustrated by the following?
    - (a) $6 + (-6) = 0$
    - (b) $4 + 0 = 4$
    - (c) $-5 = 0 - 5$
    - (d) $-(-(-4)) = -4$
    - (e) 2 is in N , or -2 is in N .
3. Show that the operation of subtraction is not commutative.
4. Is the operation of subtraction associative? If it is, prove that it is. If it is not, show that it is not by giving an example.
5. Prove for all integers: $x = y$ if and only if $-x = -y$.
6. Prove that 0 is its own additive inverse.
7. Prove that 0 is the only integer which is its own additive inverse.
8. Prove that no natural number is the additive inverse of any natural number.

The system I contains many numbers not in N , but the new system possesses all of the E,A,M,D properties that N does. When one first encounters the system I he faces the task of learning how to work with the new numbers: how to add them, how to multiply them, etc. We shall show next that these "computation rules" are all consequences of the E,A,M,D properties. Moreover, we shall see in later sections of this chapter, and in Chapter 5, that the same thing happens with each extension of the number system: the rules for calculating with the "new" numbers all follow from the properties E,A,M,D of the "new" system, most of which carry over from the "old" system.

The "new" numbers here are 0 and the additive inverses of the natural numbers. Addition of 0 and any element of I is covered by property A_4 :

$$a + 0 = a .$$

Multiplication by 0 is even simpler.

Theorem 1-4b: $a \cdot 0 = 0$, for arbitrary a in I.

Proof: By virtue of property M_1 (Closure for multiplication), $a \cdot 0$ is some member of I. Our object is to show that it is the number 0. A_5 (Subtraction) implies that for each element b in I, there is exactly one number in I satisfying the equation

$$b + x = b\ ;$$

moreover, by A_4 (Additive Identity) that number is 0. Now $a \cdot 0$ is a member of I, so the only member of I satisfying

$$a \cdot 0 + x = a \cdot 0$$

is 0. We show that $a \cdot 0$ satisfies this equation (from which it follows that $a \cdot 0$ and 0 are equal):

$$\begin{aligned} a \cdot 0 + a \cdot 0 &= a(0 + 0) && \text{[Dist.} \\ &= a \cdot 0 && \text{[Add. Ident.} \end{aligned}$$

Thus $a \cdot 0$ is a solution of $a \cdot 0 + x = a \cdot 0$. Therefore $a \cdot 0 = 0$.

We turn next to the addition and then to the multiplication of additive inverses.

Theorem 1-4c: $a + (-b) = a - b$ for arbitrary $a, b,$ in I.

Proof: Property A_5 (Subtraction) and Definition 1-4a assert that $a - b$ is the only member of I satisfying the equation

$$b + x = a\ .$$

Since this equation has only one solution in I we must conclude that $a - b$ and $a + (-b)$ are equal if we can show that $a + (-b)$ satisfies the equation $b + x = a$. But this is easy!

$$\begin{aligned} b + (a + (-b)) &= b + ((-b) + a) && [\text{Comm.} \\ &= (b + (-b)) + a && [\text{Assoc.} \\ &= 0 + a && [\text{Add. Inverse} \\ &= a && [\text{Add. Identity} \end{aligned}$$

Theorem 1-4d: $(-a) + (-b) = -(a + b)$ for arbitrary a , b in I .

We leave the proof of Theorem 1-4d as an exercise.

Theorem 1-4e: $a(-1) = -a$, for arbitrary a in I .

(This theorem asserts that the product of any number and the additive inverse of 1 is the additive inverse of the given number. This theorem and Theorem 1-4b often strike one as rather remarkable on first encounter. They are remarkable because they relate notions which are "additive" (additive identity and additive inverse, respectively) with the multiplication operation and *its* identity. Note that in each proof it is the distributive property which plays a prominent role. This is the only one of our basic properties concerned with both of these operations.)

Proof: Since $-a$ is the only integer satisfying

$$a + x = 0 ,$$

it will be sufficient to prove that $a(-1)$ satisfies this equation. Now

$$\begin{aligned} a + a(-1) &= a \cdot 1 + a(-1) && [\text{Mult. Ident.} \\ &= a(1 + (-1)) && [\text{Dist.} \\ &= a \cdot 0 && [\text{Add. Inverse} \\ &= 0 && [\text{Th. 1-4b} \end{aligned}$$

Theorem 1-4f: $(-a)b = -(ab)$, for arbitrary a , b in I .

Theorem 1-4g: $(-a)(-b) = ab$, for arbitrary a, b in I .

We leave the proofs of Theorems 1-4f, 1-4g as exercises.

In Sections 1-2 and 1-3 we discussed the cancellation properties and their converses for the system N . We found in Section 1-3 that $\underline{C}_1$, $\underline{C}_2(N)$, $\underline{C}_3$, $\underline{C}_4(N)$ can be deduced from the $\underline{E},\underline{A},\underline{M},\underline{O}$ properties of N . We now consider cancellation properties for the system I .

We shall see in Theorem 1-4h that $\underline{C}_1$, the converse of $\underline{E}_5$ (Addition), holds in I . We shall even see that in I it is easier to prove $\underline{C}_1$ than it is in N . In particular, we can prove it using the notion of additive inverse without recourse to any order properties. (In N , we could not make such a proof for we have no additive inverses in N .)

However the converse of $\underline{E}_6$ (Multiplication) is not true in I . This is so because of Theorem 1-4b $(a \cdot 0 = 0)$: if we allow $c = 0$ we cannot possibly conclude from $ac = bc$, that $a = b$. We shall see (Theorem 1-4i) that, except for this single value of c , we do have a multiplicative cancellation "law".

Theorem 1-4h: ($\underline{C}_1$) If $a + c = b + c$, then $a = b$.

Proof: If $a + c = b + c$, then

$(a + c) + (-c) = (b + c) + (-c)$ [$\underline{E}_5$

so $a + (c + (-c)) = b + (c + (-c))$ [Assoc.

and $a + 0 = b + 0$ [Add. Inverse

hence $a = b$. [Add. Ident.

Theorem 1-4i: ($\underline{C}_2$(I)) If $ac = bc$ and $c \neq 0$, then $a = b$.

The proof of $\underline{C}_2$(I) is more involved than that for $\underline{C}_1$. It is possible however to make a proof which uses only the $\underline{E}$,$\underline{A}$,$\underline{M}$ properties of I and the fact that $\underline{C}_2$(N) is valid in N . (See Exercise 1-4b, *5.)

As before, we may combine $\underline{C}_1$ and $\underline{C}_2$(I) with their respective converses $\underline{E}_5$ and $\underline{E}_6$ to get in I :

$\underline{EC}_1$: $a = b$ if and only if $a + c = b + c$

$\underline{EC}_2$: For $c \neq 0$,

$a = b$ if and only if $ac = bc$.

Since $0 \cdot c = 0$ for every c in I , we have the very important special case of $\underline{EC}_2$ obtained by taking $b = 0$:

For $c \neq 0$, $ac = 0$ if and only if $a = 0$.

Equivalently,

$ac = 0$ if and only if $a = 0$ or $c = 0$.

Exercises 1-4b

1. Perform the indicated operations using natural numbers and list the properties or theorems used.

(a)	$1 + (-2)$	(f)	$(-2) \cdot (-7)$
(b)	$12 - (-4)$	(g)	$3(a + 2) - 4(a + 2)$
(c)	$(-8) - (-7)$	(h)	$-5 \cdot (6) \cdot (-3)$
(d)	$(-5) + 7$	(i)	$4(5a)(0)$
(e)	$(-4) \cdot (5)$	(j)	$-(2a - 3) + 4(3 - 2a)$

2. Prove the following statements for all integers.

   (a) $-(x - y) = y - x$

   (b) $(-x) + (-y) = -(x + y)$. (Theorem 1-4d)

   (c) $(-x)y = -(xy)$. (Theorem 1-4f)

   (d) $(-x)(-y) = xy$. (Theorem 1-4g)

3. State and prove a "distributive" law relating the operations of multiplication and subtraction.

4. Solve each of the following equations in the system I , listing the E,A,M,D,C properties used.

(a) $5x - 3 = 12$ (d) $2(6z + 2) + 3 = 12 - 3(2z - 1)$
(b) $3y + 4 = 2y - 18$ (e) $x - 1 = x - 2$
(c) $3m - 2(7 - 2m) = 21$ (f) $100(p + 4) + 11p = 111p + 400$

*5. Prove $\underline{C}_2(I)$: If $ac = bc$ and $c \neq 0$, then $a = b$.

Show first that $ac = bc$ if and only if $-(ac) = -(bc)$, and then consider cases according as a, b, c, are natural numbers or not.

1-5. Order of the Integers.

In Section 1-3 we studied the order properties of the natural numbers. In this section we extend the order relation to the system I . We therefore face the problem of defining $a < b$ for integral a, b in such a way that our new definition agrees with the former one whenever the integers a, b are natural numbers.

Recall the criterion for $a < b$ when a, b are natural numbers: There is a natural number c such that $a + c = b$. We shall prove (Theorem 1-5a) that for integers a, b: if $a \not< b$, then either there is a natural number c such that $a + c = b$, or there is a natural number d such that $a = b + d$, but that not both of c, d can be natural numbers. With this theorem as our justification, we can then define $a < b$ for integers a, b using exactly the same words as for natural numbers (Definition 1-5a). After this we examine properties $\underline{O}_1$, $\underline{O}_2$, $\underline{O}_3$, etc., for the system I .

Theorem 1-5a: Suppose a and b are integers and $a \neq b$. Let $c = b - a$ and $d = a - b$. Then one of the integers c, d is a natural number and the other is not.

Proof: We show first that $-c = a - b$:

$$\begin{aligned} -c &= -(b - a) && [\text{Exer. 1-4a, 5} \\ &= (-1)(b + (-a)) && [\text{Ths. 1-4c, e} \\ &= (-1)b + (-1)(-a) && [\text{Dist.} \\ &= (-1)(-a) + (-1)b && [\text{Comm.} \\ &= a - b, && [\text{Ths. 1-4c, e, g} \end{aligned}$$

Thus $d = -c$. Next, because $a \neq b$, we have $c = b - a \neq 0$; and hence, by Corollary 1-4a, either c or $-c$ is a natural number. If c is a natural number, then $-c$ is not a natural number being the additive inverse of a natural number. But if $-c$ is a natural number, then c is not a natural number being the additive inverse of a natural number. Summarizing the possible cases: one of the integers c, d is a natural number and the other is not. The theorem is proved.

Definition 1-5a: If a and b are integers,

$$a < b \quad \text{means} \quad \left\{ \begin{array}{l} \text{there is a natural number } c \\ \text{such that } a + c = b . \end{array} \right.$$

It is customary to use the terms "positive" and "negative" as introduced in the following definition.

Definition 1-5b: a is positive means $0 < a$.
a is negative means $a < 0$.

It follows immediately from Definition 1-5a that every natural number is a positive integer, for $0 + a = a$ gives $0 < a$ if a is a natural number. On the other hand, the additive inverses of all the natural numbers are negative integers because $(-a) + a = 0$ gives $-a < 0$ if a is a natural number.

These observations permit us to recast Definition 1-5a in the following equivalent form.

Definition 1-5c: If a and b are integers,

$$a < b \qquad \text{if and only if} \qquad 0 < b - a .$$

In preparation for our discussion of the order properties O_1, O_2, O_3, etc., of I we prove three useful theorems about products of integers.

Theorem 1-5b: If $0 < a$ and $0 < b$, then $0 < ab$.

Theorem 1-5c: If $0 < a$ and $b < 0$, then $ab < 0$.

Theorem 1-5d: If $a < 0$ and $b < 0$, then $0 < ab$.

These theorems all follow from Corollary 1-4a, the multiplicative closure of N, and the fact that positive integers are natural numbers.

Proof of Theorem 1-5b: If $0 < a$ and $0 < b$, then a and b are natural numbers. Hence ab is a natural number, and $0 < ab$.

Proof of Theorem 1-5c: If $0 < a$ and $b < 0$, then a and $-b$ are natural numbers. Hence $a(-b)$, or $-(ab)$, is a natural number, and $ab < 0$.

We leave the proof of Theorem 1-5d as an exercise.

Now for $\underline{O}_1$, $\underline{O}_2$, $\underline{O}_3$, etc. in I .

The trichotomy property of the order relation in system I is a rephrasing of Theorem 1-5a:

$\underline{O}_1$ (Trichotomy) If a and b are integers, exactly one following relations holds:

$$a = b , \qquad a < b , \qquad b < a .$$

The other basic order properties of N have their counterparts in I . $\underline{O}_2$ and $\underline{O}_3$ are identical in N and I . $\underline{O}_4$ is quite different.

$\underline{O}_2$ (Transitivity) If $a < b$ and $b < c$, then $a < c$.

$\underline{O}_3$ (Addition) If $a < b$, then $a + c < b + c$.

$\underline{O}_4$ (Multiplication) If $a < b$ and $0 < c$, then $ac < bc$; but if $a < b$ and $c < 0$, then $bc < ac$.

These properties are consequences of the Definitions 1-5a, 1-5c, Theorems 1-5a, b, c, d and properties of N . Their proofs, being straightforward, are omitted except for $\underline{O}_4$ which deserves particular discussion.

Proof of $\underline{O}_4$: (i) If $a < b$ and $0 < c$, then $ac < bc$. By Definition 1-5c, $a < b$ means $0 < b - a$. If also $0 < c$, Theorem 1-5b gives $0 < (b - a)c$ or $0 < bc - ac$. By $\underline{O}_3$ (Addition) we get $ac < bc$.

(ii) If $a < b$ and $c < 0$, then $bc < ac$. Again, if $a < b$, then $0 < b - a$. Also, if $c < 0$, then $0 < -c$. Hence $0 < (b - a)(-c)$, so $0 < ac - bc$ and $bc < ac$.

As we did for N , in I we define

$b > a$ means $a < b$

$a \leq b$ means $a < b$ or $a = b$

$a < b < c$ means $a < b$ and $b < c$

Exercises 1-5a

1. Use the symbol "<" to form true statements of order for the following integer pairs:
   - (a) 1 and -2
   - (b) -7 and -8
   - (c) -2 and 0
   - (d) x and $-x$ if $x > 0$
   - (e) $(x - y)$ and $(y - x)$ if $y > x$
   - (f) $2x$ and $-3x$ if $x < 0$
2. Prove for arbitrary integers x , y , z , w :
   - (a) If $x < y$ and $y < z$, then $x < z$. (Property $\underline{O}_2$)
   - (b) If $x < y$, then $x + z < y + z$. (Property $\underline{O}_3$)
   - (c) If $x < 0$ and $y < 0$, then $0 < xy$. (Theorem 1-5d)
   - (d) If $0 < x$ and $y < 0$, then $x > y$.
   - (e) If $x < y$, then $x - z < y - z$.
   - (f) If $x < y$, then $y - x > 0$.
   - (g) $x > 0$ if and only if $-x < 0$.
   - (h) $0 < -x$ if and only if $x < 0$.
   - (i) If $xy < yw$, then $y(w - x)$ is a natural number.
   - (j) If $x < y$ and $w > z$, then $x - w < y - z$.

In Sections 1-2, 1-3, 1-4 we discussed the cancellation properties which involve equality in the systems N and I . In Section 1-3 we discussed the cancellation properties $\underline{C}_3$, $\underline{C}_4(N)$ involving inequality in the system N . We now look at the corresponding properties in the system I . $\underline{C}_3$ has the same wording in I as it has in N :

$\underline{C}_3$ If $a + c < b + c$, then $a < b$.

It can be proved using $\underline{O}_3$ (Addition) by adding $-c$ to both members.

Recall that in N , $\underline{C}_4$ was proved using $\underline{O}_1$ (Trichotomy) and $\underline{O}_4(N)$ (Multiplication). Since property $\underline{O}_4$ (Multiplication) has a new form in I , we expect to find that $\underline{C}_4$ is also different from its mate in N . Indeed we can prove, using the same strategy as for $\underline{C}_4(N)$, that in I , $\underline{C}_4$ has the form

$\underline{C_4}$ For $0 < c$,

If $ac < bc$, then $a < b$;

but for $c < 0$,

If $ac < bc$, then $b < a$.

In I , therefore, the compound statements $\underline{OC_1}$, $\underline{OC_2}$, $\underline{EOC_1}$, $\underline{EOC_2}$, are

$\underline{OC_1}$ $a < b$ if and only if $a + c < b + c$

$\underline{OC_2}$ For $0 < c$,

$a < b$ if and only if $ac < bc$;

but for $c < 0$,

$a < b$ if and only if $bc < ac$.

$\underline{EOC_1}$ $a \leq b$ if and only if $a + c \leq b + c$

$\underline{EOC_2}$ For $0 < c$,

$a \leq b$ if and only if $ac \leq bc$;

but for $c < 0$,

$a \leq b$ if and only if $bc \leq ac$.

Just as not all integers are positive, not all additive inverses of integers are negative. Indeed it follows from Corollary 1-4a and the remarks following Definition 1-5b that

a is positive if and only if -a is negative

and

-a is positive if and only if a is negative.

Unless a is 0 , we know then that one or other of the numbers a,-a is positive and the other one is negative. Often it is useful to speak of the one which is positive without knowing which it is; and similarly for the one which is negative. For this reason we define the "absolute value" of a number as follows.

Definition 1-5d: By the absolute value, $|a|$, of the integer a, we mean

$$|a| = a \text{ if } 0 \leq a ,$$
$$|a| = -a \text{ if } a < 0 .$$

Note in particular that $|a| = 0$ if and only if $a = 0$, and that if $a \neq 0$, we always have $|a| > 0$. Thus if $a \neq 0$, $|a|$ is the positive number in the pair $a,-a$ and $-|a|$ is the negative number in the pair $a,-a$.

We prove two theorems about absolute values. The first is little more than a restatement of the remarks in the previous paragraph. The second gives us an expression for the absolute value of a product. We shall return to the subject of absolute values in Section 1-7.

Theorem 1-5e: $-|a| \leq a \leq |a|$, for arbitrary a.

Proof: There are two possibilities: (i) $0 \leq a$, (ii) $a < 0$. In the first case $a = |a|$ and since $-|a| \leq 0$ (where we have equality only for $a = 0$) we have

$$-|a| \leq 0 \leq a = |a| .$$

In the second case $a = -|a|$, and since $a \neq 0$ we have $0 < |a|$ and hence

$$-|a| = a < 0 < |a| .$$

The statement given in the theorem is an understatement comprising both of these cases.

Theorem 1-5f: $|ab| = |a||b|$ for arbitrary a, b.

Proof: Again we consider cases (i) $0 \leq a$, $0 \leq b$, (ii) $0 \leq a$, $b < 0$, (iii) $a < 0$, $0 \leq b$, (iv) $a < 0$, $b < 0$.

For case (i) $a = |a|$, $b = |b|$ and since $0 \leq ab$, $ab = |ab|$. Thus $|ab| = ab = |a||b|$. For case (ii) $a = |a|$, $-b = |b|$ and since $ab \leq 0$, $|ab| = -(ab)$. Then $|ab| = -(ab) = a(-b) = |a||b|$. The other cases are entirely similar.

Example 1-5: Find all integral solutions of the inequality

$$|x + 1| \leq 2 .$$

Solution: We split the problem into two cases:

(i) $0 \leq x + 1$, (ii) $x + 1 < 0$.

Case (i): For $0 \leq x + 1$, we have $|x + 1| = x + 1$. Now

$$0 \leq x + 1 \qquad \text{and} \qquad |x + 1| \leq 2$$

if and only if

$$0 \leq x + 1 \leq 2$$

if and only if

$$-1 \leq x \leq 1 . \qquad [\underline{EOC}_1$$

Case (ii): For $x + 1 < 0$, we have $|x + 1| = -(x + 1)$ and $0 < -(x + 1)$, so

$$x + 1 < 0 \qquad \text{and} \qquad |x + 1| \leq 2$$

if and only if

$$0 < -(x + 1) \leq 2$$

if and only if

$$-2 \leq x + 1 < 0 \qquad [\underline{EOC}_2$$

if and only if

$$-3 \leq x < -1 . \qquad [\underline{EOC}_1$$

Combining these cases, we have

$$|x + 1| \leq 2 \quad \text{if and only if} \quad -3 \leq x \leq 1 .$$

Thus the set of solutions is $\{-3 , -2 , -1 , 0 , 1\}$.

Exercises 1-5b

1. Solve the following inequalities:
   (a) $5m - 2 < 13$, for m in N
   (b) $5m - 2 < 13$, for m in I
   (c) $4z - 7 < 2z + 3$, for z in N
   (d) $4z - 7 < 2z + 3$, for z in I
   (e) $4x - 1 < 2(x + 1)$, for x in N
   (f) $4y - 1 < 2(y + 1)$, for y in I
   (g) $5 < 7p - 2 < 12$, for p in I
   (h) $y - 1 \leq 2y - 3 \leq y + 1$, for y in I

2. Solve the following where all letters represent integers:
   (a) $|p| = 3$
   (b) $|c| < 4$
   (c) $|x + 4| < -1$
   (d) $|2m + 1| = 3$
   (e) $|4y - 1| -7 = 0$
   (f) $|x + 3| \leq 7$
   (g) $|x - 5| < 3$
   (h) $6 \geq |3 - x|$
   (i) $5 + |x + 6| < 8$

3. Prove that the following statements are true for arbitrary x in I :
   (a) If $0 < x$, then $0 < x^2$
   (b) If $1 < x$, then $x < x^2$
   (c) If $1 < x$, then $-x < x^2$
   (d) If $x < -1$, then $-x^2 < x$

4. Finish the proof of Theorem 1-5f.

5. Prove the following theorem: $|x + y| \leq |x| + |y|$
   Use an argument by cases as in Theorem 1-5f. (For reasons which will appear in Chapter 5, this inequality is called the "triangle inequality".)

The Archimedean property $\underline{O}_5$ (Section 1-3) is valid in I if we replace "natural numbers" by "positive integer". The well order property $\underline{O}_6(N)$, however does not hold in I. For example, the set of negative integers does not have a minimal member.

Both N and I, however, are so-called "discrete" systems. (Q and R are not.) In saying that N and I are discrete we mean that the integers are not "too close together"--more precisely, if a and b are distinct integers $|a - b|$ cannot be less than 1. This fact follows from the well order property of N for if a and b are distinct integers $|a - b|$ is a natural number, and hence its minimal value is a natural number. Since no natural number is less than 1, $|a - b|$ must be at least 1, no matter what integers a,b may represent.

LIST OF BASIC PROPERTIES OF THE SYSTEM OF INTEGERS

For arbitrary a, b, c in I:

$\underline{E}_1$ (Dichotomy) Either $a = b$ or $a \neq b$.

$\underline{E}_2$ (Reflexivity) $a = a$.

$\underline{E}_3$ (Symmetry) If $a = b$, then $b = a$.

$\underline{E}_4$ (Transitivity) If $a = b$ and $b = c$, then $a = c$.

$\underline{E}_5$ (Addition) If $a = b$, then $a + c = b + c$.

$\underline{E}_6$ (Multiplication) If $a = b$, then $ac = bc$.

$\underline{A}_1$ (Closure) If a and b are integers, $a + b$ is an integer.

$\underline{A}_2$ (Commutativity) $a + b = b + a$.

$\underline{A}_3$ (Associativity) $a + (b + c) = (a + b) + c$.

$\underline{A}_4$ (Additive Identity) $0 + a = a + 0 = a$.

$\underline{A}_5$ (Subtraction) If a and b are given integers, there is exactly one integer c such that $a + c = b$.

$\underline{M}_1$ (Closure) If a and b are integers, ab is an integer.

$\underline{M}_2$ (Commutativity) $ab = ba$.

$\underline{M}_3$ (Associativity) $a(bc) = (ab)c$.

$\underline{M}_4$ (Multiplicative Identity) $1 \cdot a = a \cdot 1 = a$.

$\underline{D}$ (Distributivity) $a(b + c) = ab + ac$.

$\underline{O}_1$ (Trichotomy) If a and b are integers, exactly one of the following hold:

$$a = b , \qquad a < b , \qquad b < a .$$

$\underline{O}_2$ (Transitivity) If $a < b$ and $b < c$, then $a < c$.

$\underline{O}_3$ (Addition) If $a < b$, then $a + c < b + c$.

$\underline{O}_4$ (Multiplication) If $a < b$ and $0 < c$, then $ac < bc$; but if $a < b$ and $c < 0$, then $bc < ac$.

$\underline{O}_5$ (Archimedes) If a and b are positive integers, there is a positive integer n such that $na > b$.

$\underline{O}_6$ (Discrete) If a and b are integers and $a < b$, then $1 \leq b - a$.

DEFINITIONS FOR THE SYSTEM OF INTEGERS

In the following general statements, a, b represent arbitrary members of I :

Definition 1-5a: $a < b$ if and only if there is a c in N such that $a + c = b$.

Definition 1-5b: a is positive means $0 < a$. a is negative means $a < 0$.

Definition 1-5c: $a < b$ if and only if $0 < b - a$.

Definition 1-5d: $|a| = a$, if $0 \leq a$; $|a| = -a$, if $a < 0$.

SOME THEOREMS OF THE SYSTEM OF INTEGERS

In the following general statements, a, b, c represent arbitrary members of I :

$-(-a) = a$ (Theorem 1-4a)

If $a \neq 0$, then either a is in N , or $-a$ is in N (Corollary 1-4a)

$a = b$ if and only if $-a = -b$ (Exercise 1-4a, Part 5)

$a \cdot 0 = 0$ (Theorem 1-4b)

$a + (-b) = a - b$ (Theorem 1-4c)

$(-a) + (-b) = -(a + b)$ (Theorem 1-4d)

$a(-1) = -a$ (Theorem 1-4e)

$(-a)(b) = -(ab)$ (Theorem 1-4f)

$(-a)(-b) = ab$ (Theorem 1-4g)

$a = b$ if and only if $a + c = b + c$ ($\underline{EC}_1$)

For $c \neq 0$, $a = b$ if and only if $ac = bc$ ($\underline{EC}_2$)

$ab = 0$, if and only if $a = 0$ or $b = 0$ (Corollary to $\underline{EC}_2$)

$-(a - b) = b - a$ (Exercise 1-4b, Part 2a)

For $a \neq b$, if $c = b - a$ and $d = a - b$, then one of c,d is a natural number and the other is not (Theorem 1-5a)

If $0 < a$ and $0 < b$, then $0 < ab$ (Theorem 1-5b)

If $0 < a$ and $b < 0$, then $ab < 0$ (Theorem 1-5c)

If $a < 0$ and $b < 0$, then $0 < ab$ (Theorem 1-5d)

$a < b$ if and only if $a + c < b + c$ ($\underline{OC}_1$)

For $0 < c$, $a < b$ if and only if $ac < bc$; for $c < 0$, $a < b$ if and only if $bc < ac$ ($\underline{OC}_2$)

$-|a| \leq a \leq |a|$ (Theorem 1-5e)

$|ab| = |a| \cdot |b|$ (Theorem 1-5f)

$|a + b| \leq |a| + |b|$ (Exercise 1-5b, Part 5)

1-6. The Rational Number System.

The rational number system Q contains all the integers and also the "quotient" $\frac{a}{b}$ of each pair of integers a,b $(b \neq 0)$. The elements of Q are called rational numbers. In Q each equation $bx = a$, where a and b are integers, $b \neq 0$, has a solution denoted by $\frac{a}{b}$. But we can say much more! We shall see that, even if a and b, $b \neq 0$, are any given rational numbers there is a member of Q satisfying the equation $bx = a$.

First, however, we discuss equality of rational numbers, and their sums and products.

Consider pairs of equations such as

$$2x = 4 \text{ and } 6x = 12 \quad , \quad 2x = 5 \text{ and } 6x = 15 .$$

These examples illustrate the trivial fact that a given rational number satisfies more than one equation of the form $bx = a$. Since we use the symbols a,b appearing in an equation to describe its root (we write $\frac{a}{b}$ for the root of $bx = a$) we must recognize that each rational number can be described as a quotient of integers in a variety of ways.

Because I is a part of Q, our equality relation in Q must agree with the equality relation we already have in I. Let us then determine a criterion for the pair of equations

$$bx = a \text{ and } dx = c \qquad (b \neq 0 , d \neq 0)$$

to have the same solution when we suppose that that solution is a member of I. Since we are working in I, we have at our disposal the E,A,M,D,O properties of I. Using EC_2 of I, we can say

$$bx = a \quad \text{if and only if} \quad bdx = ad \quad (\text{for } d \neq 0) ,$$
$$dx = c \quad \text{if and only if} \quad bdx = bc \quad (\text{for } b \neq 0) .$$

Now if we suppose that $bx = a$ and $dx = c$ have a common solution in I, say e, then

$$bde = ad \qquad \text{and} \qquad bde = bc .$$

Hence $\underline{E_4}$ (Transitivity) gives $ad = bc$. Conversely, if we suppose that $bx = a$ and $dx = c$ have solutions in I , say e and f , respectively, and that $ad = bc$, then we have

$$bde = ad \quad , \quad ad = bc \ , \quad bdf = bc$$

and $\underline{E_4}$ gives

$$bde = bdf \ .$$

But $bd \neq 0$, so $e = f$ and the solutions of $bx = a$, $dx = c$ are the same. Summarizing: $bx = a$ and $dx = c$, $b \neq 0$, $d \neq 0$, have the same solution in I if and only if $ad = bc$.

With this clue as our guide, we now extend the equality relation to all of Q .

Definition 1-6a: If a,b,c,d are integers, $b \neq 0$, $d \neq 0$,

$$\frac{a}{b} = \frac{c}{d} \quad \text{means} \quad ad = bc \ .$$

This relation is clearly reflexive:

$$\frac{a}{b} = \frac{a}{b} \quad \text{for} \quad ab = ab \ ;$$

it is also symmetric and transitive. Thus $\underline{E_2}$, $\underline{E_3}$, $\underline{E_4}$ hold in Q .

We illustrate the definition by considering an important example.

Theorem 1-6a: If a,b,c are integers, b and c not zero,

$$\frac{ac}{bc} = \frac{a}{b} \ .$$

Proof: The definition states that

$$\frac{ac}{bc} = \frac{a}{b} \quad \text{if and only if} \quad (ac)b = (bc)a \ ,$$

and the last equality follows from the commutativity and associativity of multiplication in I .

As numerical illustrations we have:

$$\frac{16}{64} = \frac{1\cdot 16}{4\cdot 16} = \frac{1}{4} \quad , \quad \frac{0}{10} = \frac{0\cdot 10}{1\cdot 10} = \frac{0}{1} .$$

Exercises 1-6a

1. Solve each of the following equations (a,b,c,d in I):
   (a) $5x = 3$
   (b) $2x + 1 = 6$
   (c) $2(y - 1) + 2 = y - 4$
   (d) $ax + b = c \quad (a \neq 0)$
   (e) $a(x - 2) + b = cx + d \quad (a \neq c)$
2. For what value of k will the following pairs of rational numbers be equal?
   (a) $\frac{2}{3}$, $\frac{k}{9}$
   (b) $\frac{4}{k}$, $\frac{6}{9}$
   (c) $\frac{3}{5}$, $\frac{k+1}{10}$
3. Show that $\underline{E}_3$ (Symmetry) holds in Q .

Our next problem is to determine how we should add and multiply rational numbers. Again we turn to I for the clues to our general definitions. What can we say about the equations $bx = a$ satisfied by the sum and product of pairs of integers?

Suppose $b \neq 0$, $d \neq 0$, and that x_1 and x_2 are integers satisfying

$$bx = a \quad \text{and} \quad dx = c ,$$

respectively. Then

$$bx_1 = a \quad \text{and} \quad dx_2 = c,$$

and we have

$$bdx_1 + bdx_2 = ad + bc \quad \text{and} \quad (bx_1)(dx_2) = ac .$$

Hence

$$bd(x_1 + x_2) = ad + bc \quad \text{and} \quad bd(x_1x_2) = ac .$$

Thus $$x_1 + x_2 = \frac{ad + bc}{bd} \quad \text{and} \quad x_1x_2 = \frac{ac}{bd} .$$

With these clues as guide we define addition and multiplication in Q as follows:

Definition 1-6b: If a, b, c, d are integers, $b \neq 0$, $d \neq 0$,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} ,$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} .$$

Note immediately that Q is closed under addition and multiplication, ($\underline{A}_1$, $\underline{M}_1$), for if a, b, c, d are integers so are $ad + bc$, ac and bd and moreover $bd \neq 0$ if $b \neq 0$ and $d \neq 0$. It also follows that Q has all of the other $\underline{E}$,$\underline{A}$,$\underline{M}$,$\underline{D}$ properties which I has.

Addition and multiplication in Definition 1-6b are commutative and associative (A_2 , A_3 , M_2 , M_3) . For example:

$$\frac{c}{d} + \frac{a}{b} = \frac{cb + da}{db} = \frac{ad + bc}{bd} = \frac{a}{b} + \frac{c}{d} , \qquad (\underline{A}_2)$$

$$\frac{a}{b}\left(\frac{c}{d} \cdot \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{ce}{df} = \frac{a(ce)}{b(df)} = \frac{(ac)e}{(bd)f} = \left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} . \qquad (\underline{M}_3)$$

Moreover, the equations

$$x = 0 \quad \text{and} \quad 1 \cdot x = 0$$

are equivalent, hence $0 = \frac{0}{1}$. Also

$$x = 1 \quad \text{and} \quad 1 \cdot x = 1$$

are equivalent, hence $1 = \frac{1}{1}$. Thus, for $b \neq 0$,

$$\frac{a}{b} = 0 \text{ if and only if } a = 0$$

since

$$\frac{a}{b} = \frac{0}{1} \quad \text{if and only if} \quad a \cdot 1 = b \cdot 0 \ ;$$

and

$$\frac{a}{b} = 1 \quad \text{if and only if} \quad a = b$$

since.

$$\frac{a}{b} = \frac{1}{1} \quad \text{if and only if} \quad a \cdot 1 = b \cdot 1 \ .$$

Therefore, Q has the identity properties $\underline{A_4}$, $\underline{M_4}$:

$$\frac{a}{b} + 0 = \frac{a}{b} + \frac{0}{1} = \frac{a \cdot 1 + b \cdot 0}{b \cdot 1} = \frac{a}{b}\ , \ b \neq 0,$$

$$\frac{a}{b} \cdot 1 = \frac{a}{b} \cdot \frac{1}{1} = \frac{a \cdot 1}{b \cdot 1} = \frac{a}{b}\ , \ b \neq 0,$$

Q also has the subtraction property $\underline{A_5}$, for

$$\frac{a}{b} + \frac{(-a)}{b} = \frac{ab + (-a)b}{b^2} = \frac{0}{b^2} = 0\ , \ b \neq 0.$$

Since $\frac{(-a)}{b}$, $b \neq 0$, satisfies the equation $\frac{a}{b} + x = 0$, we call it the additive inverse of $\frac{a}{b}$ and write $\frac{(-a)}{b} = -\frac{a}{b}$. Also, the equations

$$1 \cdot x = -1 \quad \text{and} \quad (-1)x = 1$$

are equivalent, hence

$$\frac{-1}{1} = \frac{1}{-1} = -1\ .$$

Hence, for $b \neq 0$,

$$\frac{(-a)}{b} = \frac{(-1)a}{1 \cdot b} = \frac{-1}{1} \cdot \frac{a}{b} = \frac{1}{-1} \cdot \frac{a}{b} = \frac{1 \cdot a}{(-1)b} = \frac{a}{-b}\ ;$$

and so

$$\frac{(-a)}{b} = \frac{a}{(-b)} = -\frac{a}{b}\ .$$

Moreover, if $a \neq 0$ and $b \neq 0$, we have

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = 1 ;$$

and, in particular,

$$a \cdot \frac{1}{a} = 1 .$$

We call $\frac{1}{a}$ the multiplicative inverse (or reciprocal) of a and $\frac{b}{a}$ the multiplicative inverse (or reciprocal) of $\frac{a}{b}$. We may then say that every non-zero member of Q has a multiplicative inverse in Q. Therefore each rational number $\frac{a}{b}$, a and b integers, $b \neq 0$, may be written as the product of the integer a and the multiplicative inverse of b.

$$\frac{a}{b} = \frac{a}{1} \cdot \frac{1}{b} = a \cdot \frac{1}{b}$$

since $a = \frac{a}{1}$:

Exercises 1-6b

1. Find the following sums. All letters represent integers.
   - (a) $\frac{2}{3} + \frac{1}{5}$
   - (b) $\frac{x}{2} + \frac{y}{z}$, $z \neq 0$
   - (c) $\frac{a}{b} + 2$, $b \neq 0$
   - (d) $\frac{x}{y+z} + \frac{x}{y}$, $y + z \neq 0$ and $y \neq 0$
   - (e) $\frac{2p+1}{3} + \frac{p-2}{4}$
   - (f) $\frac{3}{c} + \frac{4}{-c}$, $c \neq 0$
2. Find the following products. All letters represent integers.
   - (a) $\frac{2}{3} \cdot \frac{2}{5}$
   - (b) $\frac{x}{4} \cdot \frac{y}{z}$, $z \neq 0$
   - (c) $\frac{a}{b} \cdot 2$, $b \neq 0$
   - (d) $\frac{x}{y+z} \cdot \frac{x}{y}$, $y + z \neq 0$ and $y \neq 0$
   - (e) $\frac{2p+1}{3} \cdot \frac{p-2}{4} \cdot 0$

3. Prove $\underline{E}_5$ (Addition) for Q .
4. Prove $\underline{E}_6$ (Multiplication) for Q .
5. Prove that addition is associative in Q . ($\underline{A}_3$)
6. Prove that multiplication is commutative in Q . ($\underline{M}_2$)
7. Prove that $\frac{-a}{a} = -1$, when $a \neq 0$, a in I .
8. Prove that $\frac{na}{n} = a$ for a , n in I and $n \neq 0$.
9. Prove that for $a \neq 0$, $\frac{b}{a} = 0$ if an only if $b = 0$.

The cancellation properties $\underline{C}_1$ (Addition), $\underline{C}_2$ (Multiplication) both follow from the $\underline{E},\underline{A},\underline{M}$ properties we have found are valid in Q .

For example, if $\frac{a}{b} \cdot \frac{c}{d} = \frac{e}{f} \cdot \frac{c}{d}$ and $\frac{c}{d} \neq 0$, b, d, $f \neq 0$

then $$(\frac{a}{b} \cdot \frac{c}{d})\frac{d}{c} = (\frac{e}{f} \cdot \frac{c}{d})\frac{d}{c}$$

so $$\frac{a}{b}(\frac{c}{d} \cdot \frac{d}{c}) = \frac{e}{f}(\frac{c}{d} \cdot \frac{d}{c})$$

and $$\frac{a}{b} = \frac{e}{f} . \qquad (\underline{C}_2)$$

Now let us consider an equation $bx = a$, where a and b are rational numbers, $b \neq 0$. Write $a = \frac{a_1}{a_2}$, $b = \frac{b_1}{b_2}$ where a_1 , a_2 , b_1 , b_2 are integers, $b_1 \neq 0$, $b_2 \neq 0$, $a_2 \neq 0$. Then $bx = a$ is the same as

$$\frac{b_1}{b_2}x = \frac{a_1}{a_2} .$$

Hence $bx = a$

if and only if $a_2b_2\frac{b_1}{b_2}x = a_2b_2(\frac{a_1}{a_2})$

if and only if $a_2b_1x = a_1b_2$

if and only if $cx = d$

where $c = a_2b_1$, $d = a_1b_2$ are integers and $c \neq 0$ since $a_2 \neq 0$, $b_1 \neq 0$. Therefore each equation $bx = a$ with rational a, b, $b \neq 0$ has a solution in Q ; it is the solution of an equation $cx = d$ with c, d integers, $c \neq 0$.

This proves that we have a new M property for Q .

$\underline{M}_5$ (Division) Corresponding to each pair a,b of rational numbers, $b \neq 0$, there is exactly one rational number c such that $bc = a$.

Example 1-6a: Solve the equation $\frac{1}{2}x = \frac{1}{3}$ in Q .

Solutions: Using the method given above we multipy each member by 6 :

$\frac{1}{2}x = \frac{1}{3}$ if and only if $6(\frac{1}{2}x) = 6(\frac{1}{3})$ [$\underline{EC}_2$

if an only if $3x = 2$ [Th. 1-6a

if and only if $x = \frac{2}{3}$ [$\underline{M}_5$

However, we may shorten the work if we merely multiply by 2 , the reciprocal of $\frac{1}{2}$:

$\frac{1}{2}x = \frac{1}{3}$ if and only if $2(\frac{1}{2}x) = 2(\frac{1}{3})$ [$\underline{EC}_2$

if and only if $x = \frac{2}{3}$ [Mult. Inverse

The alternative solution for Example 1-6b suggests a general method which gives a second proof that $\underline{M}_5$ holds in Q :

If a and b are members of Q , $b \neq 0$, then

$$bx = a \quad \text{if and only if} \quad \frac{1}{b}(bx) = \frac{1}{b} \cdot a \qquad [\underline{EC}_2$$

$$\text{if and only if} \quad x = \frac{1}{b} \cdot a \quad [\text{Mult. Inv.}$$

but, as $\frac{1}{b}$ and a are members of Q , so is their product, by $\underline{M}_1$ (Closure). Thus $bx = a$ has exactly one solution in Q . It is $\frac{1}{b} \cdot a$, or $\frac{a}{b}$.

Example 1-6b: Solve $2x + \frac{3}{5} = \frac{1}{3}$ in Q .

Solutions: Our two methods give (writing "iff" for "if and only if")

	$15(2x + \frac{3}{5}) = 15 \cdot \frac{1}{3}$		$\frac{1}{2}(2x) = \frac{1}{2}(\frac{1}{3} - \frac{3}{5})$
iff	$30x + 9 = 5$	iff	$x = \frac{1}{2} \cdot \frac{5 - 9}{15}$
iff	$x = \frac{5 - 9}{30} = \frac{-2}{15}$	iff	$x = \frac{1}{2} \cdot \frac{-4}{15} = \frac{-2}{15}$

Some people prefer the first method since it immediately converts the problem into one involving integral coefficients. The rational numbers reappear only at the end.

The restriction $b \neq Q$ in $\underline{M}_5$ (and in the preceding discussion) merits comment. We saw that the $\underline{E}$,$\underline{A}$,$\underline{M}$,$\underline{D}$ properties of I , which are all in force in Q , lead to the conclusion $b \cdot 0 = 0$ for any b . Since the properties on which this conclusion is based hold in Q , the same conclusion holds in Q . This means that an equation $0x = a$ can have a solution in Q if and only if a is 0 ; but when a is 0 , every element of

Q satisfies the equation. Hence the desire to "divide by zero", that is, to solve $0x = a$ is doomed from the beginning. Either there is no solution at all (if $a \neq 0$) or there are too many (if $a = 0$). The two fundamental criteria for any algebraic operation,

(i) that it always be possible,

and

(ii) that it determine a definite number,

are both violated in the case of "division by zero". For (i) it is not always possible, and (ii) when it is possible it does not determine a definite member of Q. Hence by no stretch of the imagination can "division by zero" be considered an algebraic operation. We are therefore obliged to exclude it from all of our subsequent discussions.

Exercises 1-6c

1. Find solutions for the following where all letters represent rational numbers, and list the properties of the rational number system used.

   (a) $\frac{2x}{3} = 4$

   (b) $3m + \frac{2}{5} = \frac{1}{3}$

   (c) $\frac{5y - 1}{4} - 1 = \frac{3}{5}$

   (d) $\frac{2(1 - w)}{w - 1} + \frac{3w}{5} = 2 \qquad w \neq 1$

   (e) $\frac{3}{2} - \frac{4(x + 1)}{3} = \frac{2(3 + x)}{3} - 2x$

2. State and prove $\underline{EC_1}$ (Addition) for Q.

3. State and prove $\underline{EC_2}$ (Multiplication) for Q.

1-7. Order of the Rationals.

In Sections 1-3 and 1-5 we studied the order relation in N and I . We now face the question of extending it to the system Q . Since I is a part of Q such an extension must agree with the relation we already have in I .

In I , as in N , given a specific pair of numbers (such as 3 and 7 , or -2 and 9) we can spot at sight which is the larger. However it is difficult to tell at a glance whether or not $\frac{17}{27}$ is "larger" than $\frac{25}{41}$. Even deciding if these numbers are equal requires some reflection. We shall see that very little more reflection is required to decide which is the larger.

In I , our definition of order hinges on the question of whether or not a difference is positive, since for integers a, b , we have $a < b$ if and only if $0 < b - a$. To frame our definition of order for Q in the same words, we first decide which quotients shall be called "positive".

No matter how we may define an order relation in Q , if it is to agree with our order relation in I (so that $0 < b^2$ if b is a non-zero member of I) and if it is to have property O_4 (Multiplication) , then, for a and b integers, $b \neq 0$, we must have

$$0 < \frac{a}{b} \quad \text{if and only if} \quad 0 < \left(\frac{a}{b}\right)b^2$$

or, since

$$\left(\frac{a}{b}\right)b^2 = ab ,$$

$$0 < \frac{a}{b} \quad \text{if and only if} \quad 0 < ab .$$

Hence it must be the case that the quotient $\frac{a}{b}$ is "positive" (i.e., greater than 0) if and only if the product ab is positive. If we cannot agree that such a quotient be positive, then there is no hope of defining an order relation in Q consistent with the order relation we already have in I .

We therefore make the following definition:

Definition 1-7a: If a and b are integers, $b \neq 0$,

$$0 < \frac{a}{b} \quad \text{means} \quad 0 < ab .$$

Since a and b are integers, ab is also an integer. Therefore this definition bases the decision that $\frac{a}{b}$ is greater than 0 on the truth of a statement concerning two integers, 0 and ab. The truth of the latter statement is determined by our theory of order in I, where we found that $0 < ab$ if and only if either $0 < a$ and $0 < b$, or $a < 0$ and $b < 0$.

Because $\frac{a}{b} = \frac{-a}{-b}$ any rational number $\frac{a}{b}$, a and b integers, $b \neq 0$, may be written as a quotient with a positive integral denominator. But, then, if $0 < b$ we have $0 < \frac{a}{b}$ if and only if $0 < a$.

Now that we have decided which rational numbers must be positive, we return to the question of defining an order relation for all of Q. Note that

$$\frac{c}{d} - \frac{a}{b} = \frac{bc - ad}{bd} ,$$

hence if $0 < b$ and $0 < d$,

$$0 < \frac{c}{d} - \frac{a}{b} \quad \text{if and only if} \quad 0 < bc - ad .$$

We therefore frame our definition as follows.

Definition 1-7b: If a, b, c, d are integers, $0 < b$, $0 < d$,

$$\frac{a}{b} < \frac{c}{d} \quad \text{means} \quad ad < bc ;$$

or equivalently,

$$\frac{a}{b} < \frac{c}{d} \quad \text{means} \quad \begin{cases} \text{there is a positive rational number } r \\ \text{such that } \frac{a}{b} + r = \frac{c}{d} . \end{cases}$$

From this definition it can be shown that

$$\frac{a}{b} > \frac{c}{d} \quad \text{if and only if} \quad ad > bc \ .$$

These inequalities should be compared with the criterion for equality of rational numbers (Section 1-6):

$$\frac{a}{b} = \frac{c}{d} \quad \text{if and only if} \quad ad = bc \ , \ b, \ d \neq 0 \ .$$

Just as in N and I , we write $r \leq s$ for $r = s$ or $r < s$, for any rational numbers r, s . Similarly for $r \geq s$.

Returning to our question regarding $\frac{17}{27}$ and $\frac{25}{41}$, we see that the answer depends on whether or not the product $17(41)$ is greater than $27(25)$.

Because our definition of order in Q is verbally the same as that in I , we may adapt the previous proofs of $\underline{O}_1$, $\underline{O}_2$, $\underline{O}_3$, $\underline{O}_4$ to fit Q :

$\underline{O}_1$ (Trichotomy) Given any pair of rational numbers r and s , exactly one of the following relations holds:

$$r = s \ , \qquad r < s \ , \qquad s < r \ .$$

$\underline{O}_2$ (Transitivity) If $r < s$ and $s < t$, then $r < t$.

$\underline{O}_3$ (Addition) If $r < s$, then $r + t < s + t$.

$\underline{O}_4$ (Multiplication) If $r < s$ and $0 < t$, then $rt < st$; but if $r < s$ and $t < 0$, then $st < rt$.

Just as in I , we have in Q the cancellation properties

$\underline{OC}_1$ $a < b$ if and only if $a + c < b + c$

$\underline{OC}_2$ For $0 < c$,

$a < b$ if and only if $ac < bc$;

but for $c < 0$,

$a < b$ if and only if $bc < ac$.

$\underline{EOC}_1$ $a \leq b$ if and only if $a + c \leq b + c$.

$\underline{EOC}_2$ For $0 < c$,

$a \leq b$ if and only if $ac \leq bc$;

but for $c < 0$,

$a \leq b$ if and only if $bc \leq ac$.

Exercises 1-7a

1. Determine the order relation for the following pairs of rational numbers.
   - (a) $\frac{2}{3}$ and $\frac{5}{7}$
   - (b) $\frac{5}{7}$ and $\frac{12}{17}$
   - (c) $\frac{x}{5}$ and $\frac{y}{4}$, where $x < y$
   - (d) $\frac{7}{x}$ and $\frac{3}{y}$, where $xy \neq 0$ and $x < y$
   - (e) $\frac{3x + 5}{3}$ and $2x + 1$, where $x < 0$
2. Arrange the following rational numbers in a chain of inequalities and justify your arrangement:

$$\frac{-37}{61} , \frac{4}{5} , 2 , \frac{-12}{20} , \frac{47}{59} , \frac{27}{13} .$$

3. Prove: $0 < \frac{1}{a}$ if and only if $0 < a$.
4. Prove: $\frac{a}{b} > \frac{c}{d}$ if and only if $ad > bc$, $b, d \neq 0$.
5. Prove $\underline{O}_2$ for Q : if $\frac{a}{b} < \frac{c}{d}$ and $\frac{c}{d} < \frac{e}{f}$, then $\frac{a}{b} < \frac{e}{f}$, where a, b, c, d are in I , and $bdf \neq 0$.

Q also has Archimedes' property:

$\underline{O}_5$ (Archimedes) If r and s are positive rational numbers and $r < s$, then there is a positive integer n such that $nr > s$.

However the system Q has a special order property which it does not share with either N or I. Both N and I are "discrete" systems, in the sense that there is a smallest positive difference between integers. Given any pair of distinct integers, if we subtract the smaller from the larger, the resulting difference is 1 or more; there is no pair of integers whose difference is both greater than 0 and less than 1.

On the other hand we can find pairs of rational numbers whose difference is positive and "as small as we like". This will follow if we prove that between any two distinct rational numbers there is a third rational number different from either of the original two. But this is easy! Given rational numbers a and b, $a \neq b$, we may take their "average" $\frac{a + b}{2}$. Suppose $a < b$. To show that $a < \frac{a + b}{2}$ and $\frac{a + b}{2} < b$. Since $a < b$, we have $2a < a + b$ and $a + b < 2b$, (by $\underline{O}_3$), so $a < \frac{a + b}{2}$ and $\frac{a + b}{2} < b$. Thus $\frac{a + b}{2}$ is between a, b. Moreover, if a and b are rational numbers, $\frac{a + b}{2}$ is also a rational number (by $\underline{A}_1$, $\underline{M}_1$).

We have just seen that between any pair of rational numbers, there is at least one other. But, then there is a rational between this new one and each of the original ones; another rational between each of these, etc. By selecting the average each time we repeatedly halve the difference; and by repeatedly halving the difference we can make it "as small as we like".

Since it is always possible to find another rational between any two rationals, we say that the rational numbers are "dense". The special order property $\underline{O}_6(Q)$ states just this:

> $\underline{O}_6(Q)$ (Density) If a and b are rational numbers, $a \neq b$, then there is a rational number c such that $a < c < b$ or $b < c < a$. Hence between any pair of distinct rational numbers, there are infinitely many rational numbers.

Exercises 1-7b

1. Using Definition 1-7b and the fact that I has Archimedes' property $\underline{O}_5$, show that Q has Archimedes' property.
2. Prove that $a < \frac{2a+b}{3} < \frac{a+2b}{3} < b$ for a and b in Q and $a < b$.
3. Prove that $a < \frac{3a+b}{4} < \frac{2a+2b}{4} < \frac{a+3b}{4} < b$ for a and b in Q and $a < b$.

We now study the solution of inequalities in Q . Because Q is not discrete our results are quite different from those we obtain in I . We review the situation in I in order to bring out this difference.

Example 1-7a: Solve $18 < 3x + 7 < 50$ for x in I .

Solution: $18 < 3x + 7 < 50$

if and only if $11 < 3x < 43$

if and only if $\frac{11}{3} < x < \frac{43}{3}$.

Now $3 < \frac{11}{3} < 4$ and $14 < \frac{43}{3} < 15$.

So the solution set is $\{4, 5, 6, \dots, 13, 14\}$.

Our strategy in attacking this problem was to convert the given inequality, $18 < 3x + 7 < 50$, into an equivalent one of the form $a < x < b$, where a and b are certain definite numbers. The final inequality displays at a glance the range of values of x satisfying the original inequality. Since this range is limited "below" (by a) and "above" (by b) there can only be a finite number of integers in this range; just which ones we determined by finding consecutive integers on each "side" of a and b. To find the solutions in Q the matter is different. Our reasoning on behalf of the assertion

$$18 < 3x + 7 < 50 \text{ if and only if } \frac{11}{3} < x < \frac{43}{3},$$

is exactly the same whether we want rational solutions or integral solutions. But what can we do to describe the solution set in Q? In I the problem is easier because there are only finitely many members in the solution set. In Q, however, between each pair of distinct members are infinitely many more members. It is futile to contemplate any list of the solutions in Q. What we can do--and it is all that we can do--is to specify the range, say $a < x < b$, containing all the solutions. Indeed every rational number which is a solution is in this range and every rational number in this range is a solution.

Example 1-7b: Solve $-1 \leq 1 - 2x < 2$ for x in Q.

Solution: $-1 \leq 1 - 2x < 2$

if and only if $-2 \leq -2x < 1$ [$\underline{EOC}_1$, $\underline{OC}_1$

if and only if $-\frac{1}{2} < x \leq 1$. [$\underline{EOC}_2$, $\underline{OC}_2$

Exercises 1-7c

Determine all rational solutions of the following inequalities:

1. $4 < 2x + 1 < 8$
2. $-2 < 3y + 5 < 2$
3. $-1 < \frac{3a + 2}{4} < 1$
4. $-1 < 2 - m < 1$
5. $1 < \frac{7 - 2x}{5} < 14$
6. $4 < 5 - 2c \leq 5$
7. $-1 \leq 6 - 3x < 1$
8. $-\frac{1}{3} < 2x - \frac{1}{4} < \frac{1}{2}$
9. $-1 \leq \frac{3 - 2x}{-4} \leq 1$
10. $-2 < \frac{5 - 7x}{-3} \leq 2$

We return to the subject of absolute values, introduced in Section 1-5, and extend the definition to Q without change.

Definition 1-7c: If a is any rational number,

$$|a| = a \quad \text{if} \quad 0 \leq a$$

$$|a| = -a \quad \text{if} \quad a < 0 .$$

And with precisely the same proofs as in I, we may show that for arbitrary a in Q,

$$-|a| \leq a \leq |a| , \qquad \text{(Theorem 1-5e)}$$

$$|ab| = |a| \cdot |b| . \qquad \text{(Theorem 1-5f)}$$

We examine some inequalities involving absolute values.

Example 1-7c: Find all solutions of $|x| < 1$, for x in Q.

Solution: We eliminate the absolute value sign by reverting to its definition and split our discussion into two cases: (i) $0 \leq x$, (ii) $x < 0$. In case (i) we have $x = |x|$, so

for $0 \leq x$, $|x| < 1$ if and only if $0 \leq x < 1$. [Def.1-7c

In case (ii) we have $x < 0$ and $|x| = -x$, so

for $x < 0$, $|x| < 1$ if and only if $x < 0$ and $-x < 1$

if and only if $-1 < x < 0$ [$\underline{OC}_2$

Combining these cases, we get

$$|x| < 1 \text{ if and only if } -1 < x < 1 .$$

We could proceed as in Example 1-7c with other such problems. However the method of arguing by cases can become rather tedious if we must use it each time we want to eliminate an absolute value sign. It is easier to suffer through the argument once or twice to prove general theorems, which we may then use later without resorting to the two cases: (i) $0 \leq x$, (ii) $x < 0$ in each problem we meet. For this reason we prove the next two theorems.

Theorem 1-7a: Suppose $0 < a$. Then

$$|x| \leq a \quad \text{if and only if} \quad -a \leq x \leq a .$$

Proof: ("Only if") We show first that if $|x| \leq a$, then $-a \leq x \leq a$. We use two cases for the proof, (i) $0 \leq x$, (ii) $x < 0$, and show that in each case

$$\text{if} \quad |x| \leq a , \quad \text{then} \quad -a \leq x \leq a .$$

Case (i): If $0 \leq x$, then $|x| = x$ and so if $|x| \leq a$, then $0 \leq x \leq a$, hence $-a \leq x \leq a$, since $-a < 0$.

Case (ii): If $x < 0$, then $|x| = -x > 0$ and so if $|x| \leq a$, then $0 < -x \leq a$, or $-a \leq x < 0$. And since $0 < a$, we can say $-a \leq x \leq a$.

For the "if" part, we prove

$$\text{If} \quad -a \leq x \leq a , \text{ then } |x| \leq a .$$

Again we use two cases: (i) $0 \leq x$, (ii) $x < 0$.

Case (i): If $0 \leq x$ and $-a \leq x \leq a$, (i.e., $-a \leq x$ and $x \leq a$) , then from $0 \leq x$ and $x \leq a$ it follows that $|x| = x \leq a$ and so $|x| \leq a$.

Case (ii): If $x < 0$ and $-a \leq x \leq a$, then $x = -|x|$, and $-a \leq -|x|$ or $|x| \leq a$.

Example 1-7d: Solve $|2x - 1| < 5$ in Q .

Solution: From Theorem 1-7a,

$|2x - 1| < 5$ if and only if $-5 \leq 2x - 1 \leq 5$ [Th.1-7a

if and only if $-4 \leq 2x \leq 6$ [$\underline{OC}_1$

if and only if $-2 \leq x \leq 3$. [$\underline{OC}_2$

Theorem 1-7b: Suppose $0 < a$. Then

$a \leq |x|$ if and only if either $x \leq -a$ or $a \leq x$.

Proof: ("Only if") Two cases: (i) $0 \leq x$, (ii) $x < 0$.

Case (i): If $0 \leq x$ and $a \leq |x|$, then $a \leq x$.

Case (ii): If $x < 0$ and $a \leq |x|$, then $a \leq -x$, or $x \leq -a$.

For the "if" part: $0 < a$ and $a \leq x$ give $0 < x$ so $x = |x|$. Therefore, if $a \leq x$, then $a \leq |x|$. Also $0 < a$ (or $-a < 0$) and $x \leq -a$ give $x < 0$ so $x = -|x|$. Therefore, if $x \leq -a$, then $-|x| \leq -a$ or $a \leq |x|$.

Example 1-7e: Solve $2 < |1 - 2x|$ in Q.

Solution: From Theorem 1-7b,

$2 < |1 - 2x|$ if and only if $1 - 2x < -2$ or $2 < 1 - 2x$

if and only if $3 < 2x$ or $2x < -1$

if and only if $\frac{3}{2} < x$ or $x < -\frac{1}{2}$

Using Theorem 1-7a, we may give a proof, devoid of case-arguments, for the so-called "triangle inequality"

$$|a + b| \leq |a| + |b| .$$

(Cf. Exercise 1-5b)

Theorem 1-7c: $|y + z| \leq |y| + |z|$.

Proof: By Theorem 1-5e, we have

$$-|y| \leq y \leq |y|$$

$$-|z| \leq z \leq |z|$$

hence, adding,

$$-(|y| + |z|) \leq y + z \leq |y| + |z| ,$$

and, using Theorem 1-7a with $x = y + z$, $a = |y| + |z|$,

$$|y + z| \leq |y| + |z| .$$

Exercises 1-7d

1. Solve the following for x in Q .
   (a) $|x + 1| < 4$
   (b) $|2x - 1| \leq 1$
   (c) $|1 - x| > 3$
   (d) $|\frac{5 - 3x}{2}| \geq -1$
   (e) $|2x - 1| < -3$
   *(f) $2 \leq |x + 1| \leq 3$

2. Prove: If $a \leq b \leq c$ and $d \leq e \leq f$, then

$$a + d \leq b + e \leq c + f .$$

LIST OF BASIC PROPERTIES
OF THE
RATIONAL NUMBER SYSTEM

For arbitrary a, b, c in Q:

E_1 (Dichotomy) Either $a = b$ or $a \neq b$.

E_2 (Reflexivity) $a = a$.

$\underline{E}_3$ (Symmetry) If $a = b$, then $b = a$.
$\underline{E}_4$ (Transitivity) If $a = b$ and $b = c$, then $a = c$.
$\underline{E}_5$ (Addition) If $a = b$, then $a + c = b + c$.
$\underline{E}_6$ (Multiplication) If $a = b$, then $ac = bc$.

$\underline{A}_1$ (Closure) $a + b$ is a rational number.
$\underline{A}_2$ (Commutativity) $a + b = b + a$.
$\underline{A}_3$ (Associativity) $a + (b + c) = (a + b) + c$.
$\underline{A}_4$ (Additive Identity) $0 + a = a + 0 = a$.
$\underline{A}_5$ (Subtraction) For each pair a and b of rational numbers, there is exactly one rational number c such that $a + c = b$.

$\underline{M}_1$ (Closure) ab is a rational number.
$\underline{M}_2$ (Commutativity) $ab = ba$.
$\underline{M}_3$ (Associativity) $a(bc) = (ab)c$.
$\underline{M}_4$ (Multiplicative Identity) $1 \cdot a = a \cdot 1 = a$.
$\underline{M}_5$ (Division) For each pair a,b of rational numbers, $b \neq 0$, there is exactly one rational number c such that $bc = a$.

$\underline{D}$ (Distributivity) $a(b + c) = ab + ac$.

$\underline{O}_1$ (Trichotomy) If a and b are rational numbers, exactly one of the following holds:

$$a = b , \qquad a < b , \qquad b < a .$$

$\underline{O}_2$ (Transitivity) If $a < b$ and $b < c$, then $a < c$.
$\underline{O}_3$ (Addition) If $a < b$, then $a + c < b + c$.
$\underline{O}_4$ (Multiplication) If $a < b$ and $0 < c$, then $ac < bc$; but if $a < b$ and $c < 0$, then $bc < ac$.
$\underline{O}_5$ (Archimedes) If a and b are positive rational numbers and $a < b$, there is a positive integer n such that $na > b$.
$\underline{O}_6$(Q) (Density) If a and b are rational numbers, $a \neq b$, then there is a rational number c such that $a < c < b$ or $b < c < a$. Hence between any pair of distinct rational numbers there are infinitely many rational numbers.

DEFINITIONS FOR THE RATIONAL NUMBER SYSTEM

Definition 1-6a: If a, b, c, and d are integers, $b \neq 0$ and $d \neq 0$, then $\frac{a}{b} = \frac{c}{d}$ if and only if $ad = bc$.

Definition 1-6b: If a, b, c, and d are integers, $b \neq 0$ and $d \neq 0$, then

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \text{ and } \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

Definition 1-7a: If a and b are integers, $b \neq 0$, then $0 < \frac{a}{b}$ if and only if $0 < ab$.

Definition 1-7b: If a, b, c, and d are integers, $0 < b$ and $0 < d$, then $\frac{a}{b} < \frac{c}{d}$ if and only if $ad < bc$.

Definition 1-7c: If a is a rational number, $|a| = a$ if $0 \leq a$; $|a| = -a$ if $a < 0$.

SOME THEOREMS OF THE RATIONAL NUMBER SYSTEM

If a, b, and c are integers, $b \neq 0$ and $c \neq 0$, then $\frac{ac}{bc} = \frac{a}{b}$. (Theorem 1-6a)

If a and x are rational numbers and $0 < a$, then $|x| \leq a$ if and only if $-a \leq x \leq a$. (Theorem 1-7a)

If a and x are rational numbers and $0 < a$, then $a \leq |x|$ if and only if either $x \leq -a$ or $a \leq x$. (Theorem 1-7b)

If x and y are rational numbers, then $|x + y| \leq |x| + |y|$. (Theorem 1-7c)

1-8. Decimal Representation of Rational Numbers.

The long division algorithm is a procedure for converting any rational number (i.e., quotient of integers) into a decimal expression of the form

$$a_0.a_1a_2a_3\ldots$$

where a_0 is an integer and a_1, a_2, a_3, ... are decimal digits (0,1,2,3,4,5,6,7,8,9). We say that a_1 is the digit in the first place (after the point), a_2 is the digit in the second place, ... , a_n is the digit in the n^{th} place.

Some rational numbers have "terminating" decimal expressions:

$\frac{1}{2} = 0.5$ $\qquad$ $\frac{1}{8} = 0.125$

$\frac{1}{4} = 0.25$ $\qquad$ $\frac{1}{10} = 0.1$

$\frac{1}{5} = 0.2$

while others have decimal expressions which do not "terminate":

$\frac{1}{3} = 0.33333.....$ $\qquad$ $\frac{1}{7} = 0.142857142857...$

$\frac{1}{6} = 0.16666.....$ $\qquad$ $\frac{1}{9} = 0.11111.....$.

By a terminating decimal expression, we mean one with no digits but 0 after some place. Although the decimal expressions for $\frac{1}{3}$, $\frac{1}{6}$, $\frac{1}{7}$, $\frac{1}{9}$ do not terminate, they are all repeating decimal expressions in the sense that at some place a block of digits appears which is repeated thereafter. In the case of $\frac{1}{3}$, the digit 3 appears in the first place and is repeated thereafter; for $\frac{1}{6}$ the digit 6 appears in the second place and is repeated thereafter. The first six places in the decimal expression for $\frac{1}{7}$ are occupied by the digits 142857 and this block of six digits is repeated thereafter.

We indicate that a block of digits repeats by overscoring it:

$$\frac{1}{3} = 0.\overline{3}$$

$$\frac{1}{6} = 0.1\overline{6}$$

$$\frac{1}{7} = 0.\overline{142857}$$

$$\frac{1}{9} = 0.\overline{1} \ .$$

Since terminating decimal expressions are those having only 0's after some place, they may also be considered "repeating", their repeated block consisting of the single digit 0 . Accordingly we shall use the description "repeating" to include "terminating" as well. With this understanding we state the following theorem.

Theorem 1-8a: Each rational number has a repeating decimal expression.

We shall prove this theorem in general below, but first we examine a numerical example for a clue to the general proof. We carry out the division algorithm to obtain the decimal expression for $\frac{11}{7}$.

$$
\begin{array}{r@{}l}
 & 1.\overline{571428} \\
7 & \overline{)11.0000000} \\
 & \underline{7} \\
 & 40 \\
 & \underline{35} \\
 & 50 \\
 & \underline{49} \\
 & 10 \\
 & \underline{7} \\
 & 30 \\
 & \underline{28} \\
 & 20 \\
 & \underline{14} \\
 & 60 \\
 & \underline{56} \\
 & 40
\end{array}
$$

We need go no further. The expression repeats since the remainder 4 has appeared twice, each time followed by a 0 "brought down" from the dividend. This means that the digits in the quotient will come again in the same order as before.

Suppose a given remainder occurs after all the non-zero digits of the dividend have been "brought down" so that this remainder is followed by a zero. If this same remainder ever occurs a second time, again it will be followed by a zero; and from this point the decimal expression of the quotient repeats.

In view of these observations, our theorem will be proved if we can show that at least one remainder must appear more than once after all the non-zero digits of the dividend have been "brought down".

But this is easy! Our divisor is a natural number, say n. All the remainders are less than n, they are natural numbers selected from the set

$$\{0, 1, 2, \ldots, n - 1\} .$$

There are n numbers in this set. Hence any list containing more than n remainders contains at least one of them twice. The division algorithm can be made to produce a list of remainders as long as we wish. If we therefore carry out the process until we have "brought down" more than n zeros, where n is the divisor we shall have a list of remainders which contains at least one repetition. Because any such repeated remainder has a zero "brought down" behind it each time, we have produced a repetition in the digits of the quotient forcing it to repeat thereafter.

Many lists of remainders repeat before we use up all of them. For example, all the remainders obtained in converting $\frac{1}{3}$ into a decimal expression are 1's.

Not only have we found that the decimal expression for a rational number must repeat, but we can say more. The number of digits in its repeating block, which is at least one, never exceeds the divisor. As a matter of fact it is always less than the divisor, for if any remainder is zero the decimal expression of the quotient terminates then and there. So if it does not terminate, then there are only $n - 1$ possibilities for each remainder. Hence the number of digits in the repeating block is at least one and at most one less than the divisor. Note that in the case of $\frac{1}{7}$, there are 6 digits in the block.

We next prove (Theorem 1-8b) that each repeating decimal expression represents a rational number. As before, we begin with examples. Let $a = 0.\overline{123}$. Then

$$10^3 a = 123.\overline{123} = 123 + 0.\overline{123} = 123 + a$$

and $$(10^3 - 1)a = 123 \quad \text{or} \quad 999a = 123 ,$$

so $$a = \frac{123}{999} .$$

Let $b = 321.052\overline{123}$. Then

$$10^3 b = 321{,}052.\overline{123} = 321{,}052 + a .$$

Thus $$b = \frac{1}{1{,}000}(321{,}052 + \frac{123}{999}) ,$$

and we can see that b is rational. We refrain from writing it as a quotient of integers.

These examples indicate how we may construct a general proof.

Theorem 1-8b: Each repeating decimal expression represents a rational number.

Proof: We divide the proof into three cases. In the first two cases, we treat decimal expressions whose "integral parts" are zero. In the first case we consider such a decimal expression which repeats "from the beginning". Let

$$b = 0.\overline{b_1 b_2 \dots b_m} .$$

Then
$$10^m b = b_1 b_2 \dots b_m . \overline{b_1 b_2 \dots b_m}$$

$$= b_1 b_2 \dots b_m + 0.\overline{b_1 b_2 \dots b_m}$$

$$= b_1 b_2 \dots b_m + b$$

so
$$(10^m - 1)b = b_1 b_2 \dots b_m ,$$

where $b_1 b_2 \dots b_m$ is an integer whose digits are b_1 , b_2 , ... , b_m . Let us call this integer c . Then

$$b = \frac{c}{10^m - 1}$$

and b , being a quotient of integers, is a rational number. This concludes the first case.

For our second case, let us suppose that we have a decimal expression whose repeating block appears first just following the n^{th} place. Let

$$a = 0.a_1 a_2 \dots a_n \overline{b_1 b_2 \dots b_m} .$$

Then
$$10^n a = a_1 a_2 \dots a_n . \overline{b_1 b_2 \dots b_m}$$

$$= a_1 a_2 \dots a_n + 0.\overline{b_1 b_2 \dots b_m} .$$

Here $a_1a_2\ldots a_n$ is some integer, and our argument in the first case tells us that $0.\overline{b_1b_2\ldots b_m}$ represents a rational number (being a decimal expression which repeats "from the beginning"). Thus 10^na is the sum of an integer and a rational and hence is rational. Since 10^n is an integer it follows that a itself is rational. This concludes the second case.

The only remaining case is that of a decimal expression of the form

$$a_0.a_1a_2\ldots a_n\overline{b_1b_2\ldots b_m},$$

where a_0 is an integer not necessarily zero. But

$$a_0.a_1a_2\ldots a_n\overline{b_1b_2\ldots b_m} = a_0 + 0.a_1a_2\ldots a_n\overline{b_1b_2\ldots b_m},$$

the first term being an integer (by hypothesis) and the second representing a rational number (by the argument in the second case). We therefore conclude that $a_0.a_1a_2\ldots a_n\overline{b_1b_2\ldots b_m}$ represents a rational number. This ends the proof.

Exercises 1-8a

1. Find a decimal expression for each of the following rational numbers:

   (a) $\frac{7}{9}$ (d) $\frac{30}{7}$

   (b) $\frac{2}{11}$ (e) $\frac{63}{51}$

   (c) $\frac{11}{16}$

2. Find the rational number represented by each of the following decimal expressions:

   (a) $0.\overline{5}$ (d) $1.29\overline{47}$

   (b) $0.\overline{16}$ (e) $3.513\overline{72}$

   (c) $0.43\overline{12}$

3. Show that Theorems 1-8a, 1-8b are converses of each other by rephrasing them in the form "If A , then B" .
4. Prove that $1.\overline{9} = 2.\overline{0}$ by the method used to prove Theorem 1-8a.
5. Obtain decimal expressions for the following rational numbers:

$$\frac{1}{10}, \frac{1}{10^2}, \frac{1}{10^3}, \frac{1}{10^4}.$$

Describe in words a decimal expression for $\frac{1}{10^n}$, where n is any natural number.

1-9. Infinite Decimal Expressions and Real Numbers.

In Section 1-8 we examined the decimal representation of rational numbers. In this section we consider the collection of all decimal expressions

$$a_0.a_1a_2\ldots a_n\ldots ,$$

where a_0 is an integer and a_1 , a_2 , ... , a_n , ... are decimal digits (0,1,2,3,4,5,6,7,8,9) . We found in Section 1-8 that such a decimal expression represents a rational number if and only if it repeats (or terminates).

Our first observation is that some decimal expressions neither repeat nor terminate. Consider, for example, the non-terminating decimal expression

$$0.101001000100001000001000000 1\ldots$$

formed using only the digits 0,1 ; after the first 1 is one 0 , after the second 1 there are two 0's , after the third 1 there are three 0's , ... , for each natural number n , there are n zeros following the n^{th} 1 . This decimal expression does not repeat since no block of zeros is as long as any other block of zeros. Variations of this pattern will produce any number of other non-repeating decimal expressions. There are many non-repeating decimal expressions whose sequence of digits cannot be described by any such simple rule, however. We shall meet some of them later. (Section 1-10.)

The real number system, which we shall call R , may be constructed with the decimal expressions playing the role of its numbers. To construct such a system, it is necessary to define relations "equality", "order", and operations "addition", "multiplication" for the decimal expressions. Having made satisfactory definitions of these relations and operations, we should be obliged to determine which of the E,A,M,D,O properties of the rational number system might be valid in R . This is no mean task. Indeed it is quite formidable. It required several thousand years of human thought to accomplish the transition from the natural number system to the real number system. This fact alone should convince anybody that the problem is not an easy one.

A real number which is not rational is called an irrational (real) number. In "calculations" involving irrational numbers, it is customary to use rational "approximations" to them. Rational approximations to irrational numbers may be obtained by truncating (or chopping off) their decimal expressions:

$$a_0.a_1a_2\ldots a_na_{n+1}\ldots \approx a_0.a_1a_2\ldots a_n \;.$$

The sign "$\approx$" is the sign for "approximate equality"; the rational decimal on the right is obtained by truncating the infinite decimal expression on the left "after the n^{th} place".

We define equality itself for irrational decimals as follows:

Definition 1-9a: Two irrational decimals are equal if FOR EVERY n the rational decimals obtained by truncating them after the n^{th} place are equal.

This definition bases our new equality relation on the equality relation we already have for the rational numbers. Using the same idea, we may extend the order relation to R :

Definition 1-9b: One irrational decimal is *less than* another if FOR SOME n the rational truncation of the first after the n^{th} place is less than the rational truncation of the second after the n^{th} place.

Let us compare these definitions. If the truncations are *not* equal for EVERY n , they must be different for SOME n . Thus if two irrational decimals are not "equal", one of them must be "less than" the other, for given a pair of rational numbers which are not equal we know that one is less than the other. Thus our definitions have been constructed in such a way that the properties $\underline{E}_1$ (Dichotomy) and $\underline{O}_1$ (Trichotomy) hold in R . The fact that $\underline{E}_1$, $\underline{O}_1$ hold in R follows from the fact that they hold in Q .

Using the sequences of truncated rational decimal approximations formed from pairs of irrational decimal expressions we may define "sum" and "product" in R by reducing the problem to operations already defined in Q . We omit the details and merely announce that such definitions produce a number system R with all of the $\underline{E},\underline{A},\underline{M},\underline{D},\underline{O}$ properties of Q . (We restate all these properties at the end of this section.) The logical structure of R differs from that of Q only in the fact that R has one new order property, $\underline{O}_7(R)$, in addition to all six of the order properties which Q has. This new order property is stated at the end of the list of basic properties of R . Its full significance will appear when you study advanced calculus.

LIST OF BASIC PROPERTIES OF THE REAL NUMBER SYSTEM

For arbitrary a, b, c in R :

$\underline{E}_1$ (Dichotomy) Either $a = b$ or $a \neq b$.

$\underline{E}_2$ (Reflexivity) $a = a$.

$\underline{E}_3$ (Symmetry) If $a = b$, then $b = a$.

$\underline{E}_4$ (Transitivity) If $a = b$ and $b = c$, then $a = c$.

$\underline{E}_5$ (Addition) If $a = b$, then $a + c = b + c$.

$\underline{E}_6$ (Multiplication) If $a = b$, then $ac = bc$.

$\underline{A}_1$ (Closure) $a + b$ is a real number.

$\underline{A}_2$ (Commutativity) $a + b = b + a$.

$\underline{A}_3$ (Associativity) $a + (b + c) = (a + b) + c$.

$\underline{A}_4$ (Additive Identity) $0 + a = a + 0 = a$.

$\underline{A}_5$ (Subtraction) For each pair a and b of real numbers, there is exactly one real number c such that $a + c = b$.

$\underline{M}_1$ (Closure) ab is a real number.

$\underline{M}_2$ (Commutativity) $ab = ba$.

$\underline{M}_3$ (Associativity) $a(bc) = (ab)c$.

$\underline{M}_4$ (Multiplicative Identity) $1 \cdot a = a \cdot 1 = a$.

$\underline{M}_5$ (Division) For each pair a, b of real numbers, $b \neq 0$, there is exactly one real number c such that $bc = a$.

D (Distributivity) $a(b + c) = ab + ac$.

O_1 (Trichotomy) If a and b are real numbers, exactly one of the following holds:

$a = b$, $a < b$, $b < a$.

O_2 (Transitivity) If $a < b$, and $b < c$, then $a < c$.

O_3 (Addition) If $a < b$, then $a + c < b + c$.

O_4 (Multiplication) If $a < b$ and $0 < c$, then $ac < bc$; but if $a < b$ and $c < 0$, then $bc < ac$.

O_5 (Archimedes) If a and b are positive real numbers and $a < b$, there is a positive integer n such that $na > b$.

O_6 (Density) If a and b are real numbers, $a \neq b$, then there is a real number c such that $a < c < b$ or $b < c < a$. Hence between any pair of distinct real numbers there are infinitely many real numbers.

O_7(R) If $\{a_0 , a_1 , a_2 , \dots , a_n \dots\}$ and $\{b_0 , b_1 , b_2 , \dots , b_n , \dots\}$ are two sequences of real numbers with the properties

(i) $a_0 \leq a_1 \leq a_2 \leq \dots \leq a_n \leq \dots$

(ii) $b_0 \geq b_1 \geq b_2 \geq \dots \geq b_n \geq \dots$

(iii) $a_n \leq b_n$, for every natural number n

(iv) $b_n - a_n < \frac{1}{10^n}$, for every natural number n

then there is one and only one real number c such that $a_n \leq c \leq b_n$, for every natural number n .

Exercises 1-9a

1. Arrange the following in order:

   2.1545 , $2.1547\ldots$, $2.153\ldots$, 2.1547 , $2.154\overline{56}$,

   where not all of the unwritten digits in the second and third are zeros.

2. State the property of the real numbers illustrated by each of the following statements:

   (a) $x + z \leq y$ or $y < x + z$.
   (b) If $0 < x + y$ and $x + y < z$, then $0 < z$.
   (c) $(x - y)(x + y) = (x + y)(x - y)$.
   (d) If $x + y + z = x + z + z$, then $y = z$.
   (e) If $0 < -x$, then $x < 0$.
   (f) If $x - y < x + y$, then there is a z such that $x - y < z < x + y$.
   (g) $x + y = z$ or $x + y \neq z$.
   (h) $x + y + z = x + z + y$.
   (i) $4(2y) = 8y$.
   (j) If $x + y = x - y$ and $x + y = z$, then $x - y = z$.
   (k) $(x - y) + (y - x) = 0$.
   (l) $(x + y)^2 = (x + y)x + (x + y)y$.

1-10. The Equation $x^n = a$.

As stated in Section 1-9, the real number system R has all of the algebraic properties of the rational number system Q . (It has one more order property which it does not share with Q .) Thus any algebraic operation which can be performed in either can be performed in the other. From the point of view of structure they are indistinguishable except for very deep properties of their order relations. However, from the point of view of the numbers they contain they are vastly different: R contains many numbers not in Q . Because of this it is possible to solve in R some equations which cannot be solved in Q . In Corollary 1-10a we exhibit a class of equations which cannot be solved in Q . Theorem 1-10b asserts that some of them can be solved in R . However, extensive as it is, R is not vast enough to contain solutions for all of them. Another extension of our number system is required for this; it will be made in Chapter 5.

Theorem 1-10a: If n is a natural number, if a_0 , a_1 , a_2 , ... , a_{n-1} are integers, and if $\frac{p}{q}$ is a rational number satisfying the equation

$$x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \dots + a_2x^2 + a_1x + a_0 = 0 ,$$

then $\frac{p}{q}$ is an integer.

Proof: Since any rational number may be written in "lowest terms" (Theorem 1-6a), we may suppose that p and q are integers having no common integral factor greater than 1 , and that q is positive. It follows, then, that p^n and q have no common integral factor greater than 1 . By hypothesis

$$\left(\frac{p}{q}\right)^n + a_{n-1}\left(\frac{p}{q}\right)^{n-1} + \dots + a_2\left(\frac{p}{q}\right)^2 + a_1\left(\frac{p}{q}\right) + a_0 = 0 .$$

Multiplying both sides by q^{n-1} we get

$$\frac{p^n}{q} + a_{n-1}p^{n-1} + a_{n-2}p^{n-2}q + \dots + a_1pq^{n-2} + a_0q^{n-1} = 0$$

and

$$\frac{p^n}{q} = -(a_{n-1}p^{n-1} + \dots + a_0q^{n-1}) .$$

But the expression on the right is an integer. Hence $\frac{p^n}{q}$ is an integer and so q is a factor of p^n. Since the only positive common integral factor of q and p^n is 1, $q = 1$. Therefore $\frac{p}{q}$ is an integer.

Corollary 1-10a: If a is an integer and n is a natural number, the equation $x^n = a$ has a rational solution if and only if a is the n^{th} power of an integer.

The integers which are "squares" of integers are

$$0, 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, \ldots .$$

Thus if a is any integer not in this list, there is no rational number satisfying the equation $x^2 = a$. The integers which are "cubes" of integers are

$$0, 1, 8, 27, 64, 125, \ldots ,$$

and their additive inverses. Thus if a is any natural number not in this list, there is no rational number satisfying the equation $x^3 = a$. We may make similar statements about fourth powers, fifth powers, sixth powers, etc.

It is the case however--although we shall not prove it in detail--that if a is any non-negative real number (integral, rational, or irrational) and n is any natural number, then the real number system R contains exactly one non-negative number satisfying the equation $x^n = a$. (This is Theorem 1-10b.)

Perhaps the simplest case of any interest is

$$x^2 = 2 .$$

We indicate the general lines of an argument which, by filling in some details, can be used to prove that there is a real number satisfying this equation.

We know that 2 is between 1^2 and 2^2, so we are after a number between 1 and 2. Hence its decimal expression starts out $1.\cdots$. In order to find the digit in the first place after the point, we calculate

$$1.0^2,\ 1.1^2,\ 1.2^2,\ \ldots,\ 1.9^2,\ 2.0^2$$

and find that

$$1.4^2 = 1.96 < 2 < 2.25 = 1.5^2 .$$

So the first digit after the point is 4, and our number is $1.4\cdots$. Now

$$1.41^2 = 1.9881 < 2 < 2.0164 = 1.42^2$$

and we have another digit: $1.41\cdots$. Continuing,

$$1.414^2 = 1.999396 < 2 < 2.002225 = 1.415^2$$

which gives us another digit. And so on.

The fact that this procedure can be carried out as far as we may care to carry it--no matter how far that may be--and that each step produces another digit means that this procedure produces a decimal expression and therefore describes a real number.

It remains to be seen, however, whether the square of this number is 2. Let us call this number c, and write

$$\begin{array}{lcl} a_0 = 1.4 & & b_0 = 1.5 \\ a_1 = 1.41 & & b_1 = 1.42 \\ a_2 = 1.414 & & b_2 = 1.415 \\ \cdots\cdots\cdots & \text{etc.} & \cdots\cdots\cdots \end{array}$$

for the (rational) numbers produced by our procedure. Since

$$a_n < c < b_n , \quad \text{for each natural number } n ,$$

it follows that

$$a_n^2 < c^2 < b_n^2 \text{ , for each natural number } n.$$

Now

$$a_1^2 = 1.9881 \text{ , } b_1^2 = 2.0164 \text{ , } b_1^2 - a_1^2 = 0.0283 < \frac{1}{10}$$

$$a_2^2 = 1.999396 \text{ , } b_2^2 = 2.002225 \text{ , } b_2^2 - a_2^2 = 0.002829 < \frac{1}{10^2}$$

. . . .

and it can be proved that

$$0 < b_n^2 - a_n^2 < \frac{1}{10^n} \text{ , for each natural number } n .$$

Now $\underline{O}_7(R)$ (Section 1-9) tells us there is only one real number greater than every a_n^2 and less than every b_n^2. But both 2 and c^2 are there! Hence $c^2 = 2$, and we are through.

The general proof may be carried out along similar lines. We omit it entirely and simply announce the theorem.

Theorem 1-10b: Given any natural number n and any non-negative real number a, the equation $x^n = a$ is satisfied by one and only one non-negative real number.

Definition 1-10a: The unique non-negative real number satisfying $x^n = a$, n natural, a real and non-negative, is called the non-negative n^{th} root of a and is denoted by $\sqrt[n]{a}$. In the special case $n = 2$, we write simply $\sqrt{a}$, and call $\sqrt{a}$ the non-negative square root of a.

Note that for any c in R, $0 \leq c^2$. Hence if $a < 0$, the equation $x^2 = a$ has no solution in R.

We emphasize: $\sqrt{a}$ is non-negative. It is an easy consequence of the theorems on products of additive inverses that

$$(-b)^2 = b^2 ;$$

and hence, when $0 < a$, the equation $x^2 = a$ has two solutions in R. The positive solution is the one we call $\sqrt{a}$. The other solution is the additive inverse of $\sqrt{a}$; it is $-\sqrt{a}$ and is negative. For $a = 0$, $x^2 = a$ has only one solution. It is 0. Thus $\sqrt{0} = 0$.

For any real number a, we have

$$|a| = \sqrt{a^2}$$

for if $0 \leq a$, i.e., a is non-negative, then $\sqrt{a^2} = a$; while if $a < 0$, then $-a$ and $\sqrt{a^2}$ are both positive and because their squares are equal, Theorem 1-10b tells us they are equal.

Theorem 1-10c: If a and b are non-negative real numbers, then

$$\sqrt{ab} = \sqrt{a}\,\sqrt{b} ;$$

if a is a non-negative real number and b is a positive real number, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}} .$$

Proof: $\sqrt{ab}$, $\sqrt{a}$, and $\sqrt{b}$ are, respectively, the non-negative roots of

$$x^2 = ab , \qquad x^2 = a , \qquad x^2 = b .$$

If we write

$$x_1 = \sqrt{ab} , \qquad x_2 = \sqrt{a} , \qquad x_3 = \sqrt{b} ,$$

we can say $0 \leq x_1$, $0 \leq x_2$, $0 \leq x_3$ and

$$x_1^2 = ab, \quad x_2^2 = a, \quad x_3^2 = b;$$

and we want to show that $x_1 = x_2x_3$. From $x_2^2 = a$ and $x_3^2 = b$ we get

$$x_2^2x_3^2 = ab.$$

Since

$$x_2^2x_3^2 = (x_2x_2)(x_3x_3)$$

we can say

$$x_2^2x_3^2 = (x_2x_3)(x_2x_3) = (x_2x_3)^2$$

so

$$(x_2x_3)^2 = ab.$$

This proves that the product x_2x_3 satisfies the equation

$$x^2 = ab.$$

However, $0 < x_2x_3$ since $0 < x_2$ and $0 < x_3$, and x_1 is the only non-negative solution of $x^2 = ab$. Thus $x_1 = x_2x_3$ and we have proved that $\sqrt{ab} = \sqrt{a}\sqrt{b}$. We leave the proof of the other half of the theorem as an exercise.

Example 1-10a: $\sqrt{48} = \sqrt{4^2 \cdot 3} = \sqrt{4^2}\sqrt{3} = |4|\sqrt{3} = 4\sqrt{3}$

$$\sqrt{\frac{27}{20}} = \frac{\sqrt{27}}{\sqrt{20}} = \frac{\sqrt{3^2 \cdot 3}}{\sqrt{2^2 \cdot 5}} = \frac{\sqrt{3^2} \cdot \sqrt{3}}{\sqrt{2^2} \cdot \sqrt{5}} = \frac{|3|\sqrt{3}}{|2|\sqrt{5}} = \frac{3\sqrt{3}}{2\sqrt{5}}$$

If $a \neq 0$, then $\sqrt{a} \neq 0$, so

$$\sqrt{a} = \frac{a}{\sqrt{a}} \quad \text{and} \quad \frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{a}.$$

These two equalities are each equivalent to the defining equation for $\sqrt{a}$:

$$(\sqrt{a})^2 = a, \quad 0 \leq \sqrt{a}.$$

They are useful whenever one wishes to move the factor $\sqrt{a}$ from the numerator of a fraction to its denominator, or vice versa, as Examples 1-10b, c show.

Example 1-10b: $\frac{2\sqrt{7}}{9} = \frac{2}{9}\sqrt{7} = \frac{2}{9} \cdot \frac{7}{\sqrt{7}} = \frac{14}{9\sqrt{7}}$

Example 1-10c: $\frac{17}{3\sqrt{2}} = \frac{17}{3} \cdot \frac{1}{\sqrt{2}} = \frac{17}{3} \cdot \frac{\sqrt{2}}{2} = \frac{17\sqrt{2}}{6}$

Example 1-10d: Suppose a, b, and c are rational numbers. Express the reciprocal of $a + b\sqrt{c}$ in the form $A + B\sqrt{c}$, where A and B are rational numbers, determining A and B in terms of a, b, c.

Solution: We observe that $x^2 - y^2 = (x + y)(x - y)$ and hence

$$(a + b\sqrt{c})(a - b\sqrt{c}) = a^2 - b^2c .$$

Therefore

$$\frac{1}{a + b\sqrt{c}} = \frac{a - b\sqrt{c}}{a^2 - b^2c}$$

because

$$\frac{p}{q} = \frac{r}{s} \quad \text{if and only if} \quad ps = qr .$$

Note also that

$$\frac{a - b\sqrt{c}}{a^2 - b^2c} = \frac{1}{a + b\sqrt{c}} \cdot \frac{a - b\sqrt{c}}{a - b\sqrt{c}} .$$

Thus

$$A = \frac{a}{a^2 - b^2c} , \qquad B = \frac{-b}{a^2 - b^2c} .$$

Example 1-10e: Remove the radicals from the numerator of the expression

$$\frac{\sqrt{x + h} - \sqrt{x}}{h} .$$

(This question is important in elementary calculus.)

Solution: We multiply numerator and denominator both by $\sqrt{x+h}+\sqrt{x}$:

$$\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}=\frac{1}{h}\cdot\frac{(x+h)-x}{\sqrt{x+h}+\sqrt{x}}$$

$$=\frac{1}{\sqrt{x+h}+\sqrt{x}}\,.$$

Exercises 1-10a

Express each of the following in the form $a\sqrt{b}$ where a is rational and b is a natural number without square factors:

1. $\sqrt{8}$
2. $\sqrt{75}$
3. $\sqrt{98}$
4. $\sqrt{27}$
5. $\sqrt{48}$
6. $\sqrt{72}$
7. $\sqrt{192}$
8. $\sqrt{700}$
9. $\sqrt{243}$
10. $\sqrt{288}$
11. $\sqrt{\frac{3}{4}}$
12. $\sqrt{\frac{5}{9}}$
13. $\sqrt{\frac{12}{25}}$
14. $\sqrt{\frac{4}{25}}$
15. $\sqrt{\frac{27}{4}}$
16. $\sqrt{\frac{3}{5}}$
17. $\sqrt{\frac{2}{3}}$
18. $\sqrt{\frac{4}{5}}$
19. $\sqrt{\frac{5}{11}}$
20. $\sqrt{\frac{27}{8}}$
21. $\sqrt{3}\cdot\sqrt{12}$
22. $\sqrt{2}\cdot\sqrt{50}$
23. $\sqrt{5}\cdot\sqrt{12}$
24. $(\sqrt{2})^2$
25. $\sqrt{2}\cdot\sqrt{8}$
26. $\frac{\sqrt{5}}{\sqrt{6}}$
27. $\frac{\sqrt{5}}{27}$
28. $\frac{3}{\sqrt{5}}$
29. $\frac{4}{\sqrt{8}}$
30. $\frac{3\sqrt{5}}{\sqrt{27}}$
31. $\sqrt{2}+3\sqrt{2}$
32. $3\sqrt{2}-\sqrt{8}$
33. $7\sqrt{24}-5\sqrt{54}$
34. $11\sqrt{60}-3\sqrt{375}$
35. $2\sqrt{20}+3\sqrt{405}$

Express each of the following in the form $a + b\sqrt{c}$ where a and b are rational numbers and c is a natural number without square factors.

36. $(1 + \sqrt{2})(1 - \sqrt{2})$
37. $(1 + \sqrt{2})^2$
38. $(\sqrt{2} + \sqrt{3})^2 - \sqrt{24}$
39. $(1 + \sqrt{2})(2 - \sqrt{2})$
40. $(5\sqrt{2} + 2\sqrt{3})(\sqrt{2} + 3\sqrt{3})$
41. $\dfrac{1}{1 + \sqrt{2}}$
42. $\dfrac{4}{\sqrt{3} - 1}$
43. $\dfrac{\sqrt{2} + \sqrt{3}}{\sqrt{2} - \sqrt{3}}$
44. $\dfrac{1 + \sqrt{2}}{3 - \sqrt{8}}$
45. $\dfrac{4 - 2\sqrt{3}}{5 + \sqrt{192}}$

Express each of the following numbers and their squares in the form $a + b\sqrt{c}$ where a, b, and c are *integers*.

46. $\dfrac{1}{2 + \sqrt{3}}$
47. $\dfrac{1}{3 + 2\sqrt{2}}$
48. $\dfrac{1}{5 + 2\sqrt{6}}$
49. $\dfrac{1}{7 + 4\sqrt{3}}$
50. $\dfrac{1}{8 + 3\sqrt{7}}$
51. $\dfrac{1}{9 + 4\sqrt{5}}$
52. $\dfrac{1}{17 + 12\sqrt{2}}$

1-11. Polynomials and Their Factors. (Review)

By a *polynomial in* x we mean an expression such as

$$2x + 1 ,$$

$$3x^2 - x + 2 ,$$

or

$$x^3 - x^2 + 1 ,$$

which is a sum of terms of the form

$$a \ , \ bx \ , \ cx^2 \ , \ dx^3 \ , \ \ldots \ ,$$

a, b, c, d, ... being numbers. The numbers are called the *coefficients* of the polynomial. Thus the coefficients of the polynomials written above and on the preceding page are

$$2,1 \ ; \ 3,-1,2 \ ; \ 1,-1,0,1 \ ; \text{ respectively.}$$

By a *polynomial in x and y* we mean expressions such as

$$x + y \ , \ x^3y - xy^3 \ , \ 2x + 1 \ ,$$

which are sums of terms of the form a , bx , cy , dxy , ex^2 , fy^2 , gx^2y , hxy^2 , ... ; a , b , c , ... , g , h , ... being numbers, called the coefficients of the polynomial. We may define, in a similar way, polynomials in any collection of letters.

When the numbers appearing as coefficients in a given polynomial belong to a given number system we say that the polynomial is a polynomial *over* the given number system. Often a given polynomial may be interpreted as being "over" several numbers systems. We list some examples, naming number systems containing their coefficients.

$x + 1$	N , I , Q , R
$x^2 - 1$	I , Q , R
$x^2 + \frac{1}{2}$	Q , R
$x^3 - \sqrt{3}x + 7$	R

We "add" polynomials by combining terms in accordance with the commutative, associative and distributive properties. Thus

$$\begin{aligned}(x^2 + 3x - 1) + (2x^3 - 7x^2 + x) &= 2x^3 + (x^2 - 7x^2) + (3x + x) - 1 \\ &= 2x^3 - 6x^2 + 4x - 1 ,\end{aligned}$$

$$\begin{aligned}(x^2y - xy + y^3) + (x^3 - 2x^2y + xy + 7) &= x^3 + (x^2y - 2x^2y) + y^3 + (xy - xy) + 7 \\ &= x^3 - x^2y + y^3 + 7 .\end{aligned}$$

The set of all polynomials in a given set of letters is closed under addition.

We "multiply" polynomials just as we multiply numerical expressions. For example

$$\begin{aligned}(x + y)(x^2 - y) &= (x + y)x^2 - (x + y)y \\ &= x^3 + x^2y - xy - y^2 .\end{aligned}$$

The set of all polynomials in a given set of letters is closed under multiplication.

Indeed, the set of all polynomials in a given set of letters possesses all the E,A,M,D properties of the number system I as well as $\underline{C}_1$, $\underline{C}_2$(I) ; whether they are "over" I , Q , or R . Thus all of the E,A,M,D properties listed at the end of Section 1-5 as well as $\underline{C}_1$ (Theorem 1-4h, page 29) and $\underline{C}_2$(I) (Theorem 1-4i, page 30) may also be interpreted as valid statements about polynomials if the symbols "a" , "b" , "c" , occurring in them are interpreted as polynomials instead of integers.

Corresponding to the problem of expressing the product of two or more polynomials as a polynomial, we have the "reverse" problem of resolving a given polynomial into "factors". Thus, for example,

$$x^3 - 2x^2y + xy - 2y^2 = x^2(x - 2y) + y(x - 2y)$$
$$= (x - 2y)(x^2 + y).$$

In this example, we have resolved the given polynomial "over I" for its factors are both polynomials over I. As another example, we have

$$x^3 - x^2y^2 - 2xy^2 + 2y^4 = x(x^2 - 2y^2) - y^2(x^2 - 2y^2)$$
$$= (x - y^2)(x^2 - 2y^2)$$
$$= (x - y^2)(x + \sqrt{2}y)(x - \sqrt{2}y) ,$$

Where in the second line we give a factorization over I (or Q) and in the third line, it is further resolved into a product of factors over R.

The problem of factorization is the problem of expressing polynomials in factored form. The technique used to solve factorization problems amounts to reversing the steps used in expanding products. Fortunately there are only a few general types into which most factorization problems fall. We give the names of these types and their formulas in the following list:

Common Factor: $ab + ac = a(b + c)$

Difference of Squares: $a^2 - b^2 = (a + b)(a - b)$

Binomial Product: $acx^2 + (ad + bc)x + bd = (ax + b)(cx + d)$

Binomial Square: $a^2 + 2ab + b^2 = (a + b)^2$

Sum of Cubes: $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$

Difference of Cubes: $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$

Unless a and b, b and c, or c and d themselves have common factors, each factor in the right members of these formulas cannot be factored using polynomials with real coefficients.

Each of the formulas may be proved using the properties E,A,M,D by starting with the right member and expanding it. When the formulas are used for factorization we work the other way--from left to right.

We illustrate the use of these formulas in factorization problems with several examples.

Example 1-11a: Factor $2a - 2b - ac + bc$.

Solution: The common factors of the first two and last two terms suggest the "Common Factor" type (Distributivity)

$$2a - 2b - ac + bc = 2(a - b) - c(a - b) .$$

The expression on the right now consists of two terms which have a common factor, so

$$2a - 2b - ac + bc = (a - b)(2 - c) .$$

Example 1-11b: Factor $2(a - b)^2 - 18c^2$.

Solution: Using the "Common Factor",

$$2(a - b)^2 - 18c^2 = 2[(a - b)^2 - 9c^2] .$$

The second factor in the right hand expression has the form of the "Difference of Two Squares", hence

$$\begin{aligned} 2(a - b)^2 - 18c^2 &= 2[(a - b)^2 - (3c)^2] \\ &= 2[(a - b) - 3c)][(a - b) + 3c] \\ &= 2(a - b - 3c)(a - b + 3c) . \end{aligned}$$

Exercises 1-11a

Factor each of the following polynomials over the integers.

1. $5x - 5y$
2. $-6a - 16$
3. $6p - 3q + 15r$
4. $10y - 5x + 20w - 10z$
5. $12ab + 6b - 54bc$
6. $a(x + y) + b(x + y)$
7. $x(a - b) - y(a - b)$
8. $2u(x + y) - u(x + y)$
9. $b(x - y) + (x - y)$
10. $3(a + b) - (a + b)$
11. $ax + ay + bx + by$
12. $bx - by + cx - cy$
13. $bx - by - cx + cy$
14. $3a^3 + 3a^2 - 4a - 4$
15. $4m^2 - 9n^2$
16. $a^4 - 16$
17. $7c^2 - 63$
18. $x^2 - (a - b)^2$
19. $(a + b)^2 - (c + d)^2$
20. $(x - y + 1)^2 - 1$

Example 1-11c: Factor $10x^2 + 7x - 12$.

Solution: If this polynomial can be factored, it must have the binomial product form

$$acx^2 + (ad + bc)x + bd .$$

Inspection of the polynomial to be factored shows that a, b, c, d must satisfy the conditions

$$ac = 10 , \quad ad + bc = 7 , \text{ and } bd = -12 .$$

A set of values for a, b, c, d can be chosen as

$$a = 5 , \quad c = 2 \quad \text{so that} \quad ac = 10 ,$$

$$b = 2 , \quad d = -6 \quad \text{so that} \quad bd = -12 ,$$

and then tested for the third conditon. Since

$$ad + bc = -30 + 4 = -26 \neq 7 ,$$

this set is not acceptable.

Try instead:

$$a = 5 \quad , \quad c = 2$$
$$b = 4 \quad , \quad d = -3 .$$

Then,

$$ad + bc = -15 + 8 = -7 \neq 7 ;$$

but since $15 - 8 = 7$, the set can be adjusted:

$$a = 5 \quad , \quad c = 2$$
$$b = -4 \quad , \quad d = 3$$

and

$$ad + bc = 15 + (-8) = 7 .$$

Hence,

$$10x^2 + 7x - 12 = (5x - 4)(2x + 3) .$$

Example 1-11d: Factor $4y^2 + 12y + 9$.

Solution: This polynomial has the form of a Binomial Square

$$a^2 + 2ab + b^2 ,$$

Thus

$$\begin{aligned} 4y^2 + 12y + 9 &= (2y)^2 + 2 \cdot 2y \cdot 3 + (3)^2 \\ &= (2y + 3)^2 . \end{aligned}$$

Example 1-11e: Factor $6x^2 - 4x - 12$.

Solution: Noting the common factor 2 ,

$$6x^2 - 4x - 12 = 2(3x^2 - 2x - 6) .$$

The second factor on the right can be factored if it is the Binomial Product Form

$$acx^2 + (ad + bc)x + bd .$$

A possible set of a,b,c,d values is

$$a = 3 \ , \ c = 1 \quad \text{so that} \quad ac = 3 \ ,$$

$$b = 2 \ , \ d = -3 \quad \text{so that} \quad bd = -6 \ .$$

But, $$ad + bc = -9 + 2 = -7 \neq -2 \ .$$

If other integral values of a,b,c,d are chosen so that $ac = 3$ and $bd = -6$, it will be found that none of them will satisfy the conditon $ad + bc = -2$. In this case the polynomial can not be factored over the integers. (We shall see in Chapter 4 that this polynomial can be factored over the reals).

Exercises 1-11b

Factor each of the following polynomials over the integers.

1. $x^2 + 8x + 15$
2. $w^2 - 11w + 24$
3. $3a^2 - 4a - 15$
4. $4x^2 - 5x - 6$
5. $y^2 - 10y + 25$
6. $3a^2 + 4a - 4$
7. $wx^2 - 12wx + 36w$
8. $dy^2 - 11dy + 30d$
9. $25x^2 - 30xy + 9y^2$
10. $9aw^2 + 5aw - 36a$
11. $4u^2 + 12uv + 9v^2$
12. $49z^2 + 14z + 1$
13. $cx^2 - 2cx - 8c$
14. $2 - 6a - 8a^2$
15. $9 + 6c - 8c^2$
16. $15y + 42 - 3y^2$
17. $7x - 6x^2 + 20$
18. $4a^2b^2 + 4ab + 1$
19. $a^2 + 2ab + b^2 - c^2$
20. $a^2 + b^2 + 2ab - 2a - 2b + 1$.

Example 1-11f: Factor $16x^3 - 54y^3$.

Solution: Noting the common factor 2, we have

$$16x^3 - 54y^3 = 2(8x^3 - 27y^3) \ .$$

Since $2^3 = 8$ and $3^3 = 27$, the second factor on the right can be written as a Difference of Two Cubes, so

$$a^3 - b^3 = (a - b)(a^2 + ab + b^2) \qquad \text{applies.}$$

Hence,

$$\begin{aligned} 16x^3 - 54y^3 &= 2(8x^3 - 27y^3) \\ &= 2[(2x)^3 - (3y)^3] \\ &= 2(2x - 3y)(4x^2 + 6xy + 9y^2) . \end{aligned}$$

Exercises 1-11c

Factor each of the following polynomials over the integers.

1. $c^3 + d^3$
2. $w^3 - 64$
3. $x^3 + 1$
4. $m^3 - 8u^3$
5. $27r^3 + y^3$
6. $8a^3 + x^3$
7. $r^3 - 64s^3$
8. $64 - 27x^3$
9. $ac^3 - 64a$
10. $a^3b - 125b^4$
11. $27r^3y + y$
12. $4x^3 - 32$
13. $128 + 16y^3$
14. $x^6 - y^6$
15. $m^6 + u^6$

Exercises 1-11d (Miscellaneous Exercises)

Factor each of the following polynomials over the integers.

1. $12m^2 + 8m - 15$
2. $a^3 - a^2 - a + 1$
3. $4xy^2 - 13xy$
4. $d^2 + 2dh + h^2 - f^2$
5. $2am + 3bx + 3bm + 2ax$
6. $6a^3 + 9a^3b - 12a^4b^2$
7. $3x - 3y - 5xz + 5yz$
8. $a^2 - b^2 + 4a - 4b$
9. $m^3 - 8u^3$
10. $6y^2 - yz - 12z^2$
11. $kr - ks + wr - ws$
12. $8x^4y^2 - 20x^3y^2 - 12x^2y^2$
13. $16r^3 - 54$
14. $c^2 + d^2 + 2cd - h^2$
15. $x^4 + 2x^2 + 1$
16. $ay^2 - 10ay + 25a$

17. $100 - t^4$

18. $mr - ms + pr - ps$

19. $a^2 + b^2 - c^2 + 2ab$

20. $27r^3y + y$

21. $1 + 49z^2 - 14z$

22. $5cx^2 + x^3 - 5c - x$

23. $8r^3 - 125$

24. $4x^2 - y^2 - z^2 + 2yz$

25. $a^5 - 16a$

26. $16 - x^8$

27. $4x^2 - 4y^2 + 4y - 1$

28. $x^2 - x(x + y) - 20(x + y)^2$

29. $(x + y)^2 + 3(x + y) - 28$

30. $6st - 9s^2 + r^2 - t^2 - 10r + 25$

1-12. Rational Expressions. (Review)

A rational expression is a quotient of two polynomials. Examples are

$$\frac{1}{x}, \quad \frac{x-1}{x+1}, \quad \frac{x^2 + \frac{3}{2}}{y - x}, \quad \frac{x^2 + \sqrt{2}}{xy + 2} .$$

Note that the numbers used in forming the polynomials may be any of the kinds we have studied, whether integral, rational, or real. In the name "rational expression" the adjective "rational" refers to the way the letters x,y,z appear and not to the type of numbers used.

Using the formulas introduced in Section 1-6

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd} \qquad \frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}$$

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} \qquad \frac{1}{\frac{a}{b}} = \frac{b}{a}$$

and interpreting the letters as representing polynomials we have definitions for the sum, product, difference and quotient of rational expressions. The set of rational expressions is closed under these four operations and has all the E,A,M,D properties of the number system Q .

Interpreting a,b,c to be polynomials, the formula

$$\frac{ac}{bc} = \frac{a}{b}$$

enables us to "simplify" rational expressions by removing common factors from numerator and denominator (such common factors are found using the methods of Section 1-11).

Example 1-12a: Simplify the rational expression

$$\frac{(x^2 - 6x + 9)(x^2 + 3x + 9)}{(x^3 - 27)(x - 3)} .$$

Solution:
$$\frac{(x^2 - 6x + 9)(x^2 + 3x + 9)}{(x^3 - 27)(x - 3)} = \frac{(x - 3)^2(x^2 + 3x + 9)}{(x - 3)(x^2+3x+9)(x-3)}$$

$$= \frac{(x - 3)^2}{(x - 3)^2} \cdot \frac{x^2 + 3x + 9}{x^2 + 3x + 9}$$

$$= 1 .$$

A useful version of the last formula is

$$\frac{a}{b} \cdot \frac{c}{a} = \frac{c}{b} \cdot \frac{a}{a} = \frac{c}{b}$$

which can be used to simplify products of rational expressions.

Example 1-12b: Write the product $\frac{3x + 2}{x^2 - 1} \cdot \frac{3x^2 + x - 2}{9x^2 - 4}$ as a rational expression in simplified form.

Solution:
$$\frac{3x + 2}{x^2 - 1} \cdot \frac{3x^2 + x - 2}{9x^2 - 4} = \frac{3x + 2}{(x + 1)(x - 1)} \cdot \frac{(3x-2)(x+1)}{(3x+2)(3x-2)}$$

$$= \frac{3x + 2}{3x + 2} \cdot \frac{3x - 2}{3x - 2} \cdot \frac{x + 1}{x + 1} \cdot \frac{1}{x - 1}$$

$$= \frac{1}{x - 1} .$$

The phrase "complex fraction" (or compound fraction) is used for the quotient of two rational expressions, or the quotient of a rational expression and a polynomial. Examples of these are,

$$\frac{\frac{2x}{x+1}}{\frac{x-1}{x+1}}, \frac{3c^2 - 2c - 1}{\frac{3c-1}{c-1}}, \frac{\frac{1}{x} + \frac{1}{y}}{\frac{1}{x} - \frac{1}{y}} .$$

These expressions are sometimes written more compactly by using " $\div$ " to replace the quotient bar, as

$$\frac{2x}{x+1} \div \frac{x-1}{x+1}, (3c^2 - 2c - 1) \div \frac{3c-1}{c-1}, (\frac{1}{x} + \frac{1}{y}) \div (\frac{1}{x} - \frac{1}{y}) .$$

A complex fraction can be changed to a rational expression by use of the formula $\frac{ac}{bc} = \frac{a}{b}$.

Example 1-12c: Write $\frac{\frac{3x^2 - x - 2}{x-4}}{\frac{2+3x}{x}}$ as a rational expression.

Solution: $$\frac{\frac{3x^2 - x - 2}{x-4}}{\frac{2+3x}{x}} = \frac{\frac{(3x+2)(x-1)}{x-4}}{\frac{3x+2}{x}} .$$

Then $x(x - 4)$ is selected as a new factor in numerator and denominator and

$$\frac{\frac{3x^2 - x - 2}{x-4}}{\frac{2+3x}{x}} = \frac{\frac{(3x+2)(x-1)}{x-4} \cdot x(x-4)}{\frac{3x+2}{x} \cdot x(x-4)}$$

$$= \frac{x(3x+2)(x-1)}{(3x+2)(x-4)}$$

$$= \frac{x(x-1)}{x-4} .$$

Exercises 1-12a

1. Simplify each of the following rational expressions:

(a) $\frac{m}{mu}$

(b) $\frac{a^2}{ab}$

(c) $\frac{9mnp}{12m^2p^2}$

(d) $\frac{3x - 9}{9x - 9}$

(e) $\frac{10 - 5m}{10 + 5m}$

(f) $\frac{xy + xz + yw + wz}{y + z}$

(g) $\frac{ab + 2ac - 2b - 4c}{b + 2c}$

(h) $\frac{y + yz - z - z^2}{y^2 - 2yz + z^2}$

(i) $\frac{x^2 - y^2}{x + y}$

(j) $\frac{c + d}{c^2 - d^2}$

(k) $\frac{x^2 - 25}{x^2 + 10x + 25}$

(l) $\frac{y^3 - 4y}{2y^3 + 8y^2 + 8y}$

(m) $\frac{p^2 - p - 6}{2p^2 - p - 10}$

(n) $\frac{c^2 + 3c - 10}{c^2 + 2c - 15}$

(o) $\frac{x^3 - 1}{x^3 + 4x^2 - 5x}$

(p) $\frac{a^2 + a}{a^3 + 1}$

(q) $\frac{x^2 - x - 6}{2 - x - x^2}$

(r) $\frac{a^2 - ab - 2b^2}{4b^2 - a^2}$

(s) $\frac{x^2 - y^2 - z^2 + 2yz}{x + y - z}$

(t) $\frac{x^2 + y^2 - 2xy - 4}{3x - 3y + 6}$

2. Write the product or quotient in simplified form.

(a) $\frac{8b^2}{3c^2} \cdot 24bc$

(b) $\frac{x + y}{3m - 9u} \cdot \frac{12}{x + y}$

(c) $\frac{3y}{4yz - yz} \cdot \frac{2z^2 - 4z^2}{12z}$

(d) $\frac{x^2 - 2x - 15}{x^2 - 9} \cdot \frac{x^2 - 6x + 9}{3a - ax}$

(e) $\frac{x^3}{y^3} \cdot \frac{xy - y^2}{x^2 + xy}$

(f) $\frac{x - 1}{x^2 + 1} \cdot \frac{x^2 - 1}{(x - 1)^2}$

(g) $\frac{2m - 3}{m^2 - 1} \cdot \frac{2m^2 + m - 3}{4m^2 - 9}$

(h) $\frac{3a^2 - 2ab - b^2}{a^2 - b^2} \cdot \frac{1}{3a^2 + 4ab + b^2}$

(i) $\frac{p^2 - 5p + 6}{p^2 - 4} \cdot \frac{p^2 + 11p + 18}{p^2 - 2p - 3}$

(j) $\frac{c^3 - 27d^3}{c + 3d} \cdot \frac{c^2 - 9d^2}{c^2 + 3cd + 9d^2}$

(k) $\frac{2m^2 - 8}{m^2 - 3m - 10} \cdot \frac{m^2 - 6m + 5}{m^2 - 3m + 2}$

(l) $\frac{2xy + y^2}{y^2 - x^2} \cdot \frac{x + y}{xy + 2x^2}$

(m) $\frac{a - 1}{a^2 + 1} \cdot \frac{a^2 - 1}{(a - 1)^2}$

(n) $\frac{a^2 + 4ab + 3b^2}{a^2 - b^2} \cdot \frac{a^2 + b^2 - 2ab}{a + 3b} \cdot \frac{a + b}{a - b}$

(o) $\frac{4x^2 - 5x - 6}{4x^2 + 6x - 4} \cdot \frac{12x^2 + 5x - 2}{8x^2 - 6x - 9} \cdot \frac{2x^2 + x - 6}{3x^2 - 4x + 4}$

(p) $\frac{\frac{a}{2b}}{\frac{4}{c}}$

(q) $\frac{\frac{a - b}{c}}{\frac{a + b}{a}}$

(r) $\frac{\frac{2x}{x + y}}{\frac{3y}{x - y}}$

(s) $(\frac{x + 2}{x}) \div \frac{4 - x^2}{2x^2}$

Addition (and subtraction) of rational expressions is based upon the formula

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

which can be proved in a manner similar to that used in this formula with rational numbers. In practice the two formulas

$$\frac{a}{b} = \frac{ac}{bc} \quad \text{and} \quad \frac{a}{b} + \frac{c}{b} = \frac{a + c}{b} ,$$

are used as shown in the next example.

Example 1-12d: Write $\frac{x}{x + 3} + \frac{5x^2}{x^2 - 9} - \frac{1}{x - 3}$ as a rational expression.

Solution:
$$\frac{x}{x + 3} + \frac{5x^2}{x^2 - 9} - \frac{1}{x - 3} = \frac{x}{x + 3} + \frac{5x^2}{(x+3)(x-3)} - \frac{1}{x-3}$$

$$= \frac{x(x - 3)}{(x + 3)(x - 3)} + \frac{5x^2}{(x + 3)(x - 3)} - \frac{x + 3}{(x - 3)(x + 3)}$$

$$= \frac{x(x - 3) + 5x^2 - (x + 3)}{(x + 3)(x - 3)}$$

$$= \frac{6x^2 - 4x - 3}{(x + 3)(x - 3)} .$$

In some instances the formulas

$$- \frac{a}{b} = -1 \cdot \frac{a}{b} \quad \text{and} \quad -(b - a) = a - b .$$

can be used to advantage.

Example 1-12e: Write $\frac{2a}{3a - b} - \frac{a}{b - 3a}$ as a rational expression.

Solution: Since $-(b - 3a) = 3a - b$,

$$\begin{aligned}\frac{2a}{3a - b} - \frac{a}{b - 3a} &= \frac{2a}{3a - b} - \frac{a(-1)}{(b - 3a)(-1)} \\ &= \frac{2a}{3a - b} - \frac{-a}{3a - b} \\ &= \frac{3a}{3a - b} .\end{aligned}$$

Exercises 1-12b

Write each of the following sums or differences as a rational expression.

1. $\frac{x + 2}{3} + \frac{5x - 4}{2}$
2. $\frac{2a - 1}{5} - \frac{a + 3}{5}$
3. $\frac{3}{mn} + \frac{5}{mn^2} - \frac{2}{mn}$
4. $\frac{4y + 5}{4y} - \frac{y - 6}{2}$
5. $\frac{2x - 7}{7} - \frac{x + 9}{3} + \frac{x - 2}{2}$
6. $\frac{5}{a^2 - a} + \frac{2}{a^2 + a}$
7. $\frac{x}{1 - x^2} - \frac{1}{x - 1}$
8. $\frac{p}{p - 1} - \frac{1}{p^2 - p}$
9. $\frac{x}{3x - 2y} - \frac{y}{2x + 3y}$
10. $\frac{m - 3}{m + 3} - \frac{m + 3}{3 - m} + \frac{m^2}{9 - m^2}$
11. $\frac{2a - 1}{a + 3} + \frac{a^2}{3a - 1}$
12. $\frac{c}{c + 3} + \frac{5c^2}{c^2 - 9}$
13. $\frac{7}{2x + 4y} + \frac{3}{x + 2y}$
14. $\frac{2}{xy + y^2} + \frac{3}{x^2 + xy}$
15. $\frac{3}{a + b} + \frac{2a}{(a - b)^2}$
16. $\frac{2b}{b^2 + 3b + 2} - \frac{b}{b^2 - 1}$
17. $\frac{x}{x^2 - 5x + 6} + \frac{x}{x^2 - x - 6}$
18. $\frac{m^2}{m^2 + 3m - 4} - \frac{m + 7}{m + 4}$
19. $\frac{3x}{x^2 + 4x + 4} - \frac{x^2}{x^2 + 2x + 1}$
20. $\frac{y}{y^2 - 2y + 5} + \frac{3}{y - 1}$

Exercises 1-12c (Miscellaneous Exercises)

Write each of the following as a rational expression in simplified form.

1. $\frac{3a}{5} - \frac{2}{5a}$

2. $\frac{2x}{5y} \cdot \frac{15y^3}{20}$

3. $\frac{2a - b}{3a} + \frac{2a - 3b}{2b}$

4. $\frac{x + y}{3x - 9y} \cdot \frac{12}{x + y}$

5. $\frac{x + 5}{x - 5} - \frac{x - 5}{x + 5}$

6. $\frac{3c}{4bc - bc} \cdot \frac{2b^2 - 4b^2}{12b}$

7. $\frac{1}{m + 2} \cdot \frac{1}{2 - m}$

8. $\dfrac{\frac{x^2 - xy^2}{4}}{\frac{x^2 - 2xy + y^2}{8}}$

9. $\frac{m - n}{m^2 - 1} \cdot \frac{m - 1}{m^2 - n^2}$

10. $\frac{x - 1}{2x^2 - 18} - \frac{x + 2}{3x^2 + 9x}$

11. $\frac{a^2 - 2a - 15}{a^2 - 9} \cdot \frac{a^2 - 6a + 9}{3 - a}$

12. $\frac{x^3}{y^3} \div \frac{x^2 + xy}{xy - y^2}$

13. $\frac{1}{2x^2 + 7x - 15} - \frac{1}{x^2 + 6x + 5}$

14. $\frac{4}{w^2 + w - 2} - \frac{3}{w^2 + 7w + 10}$

15. $\dfrac{\frac{4}{x} - x}{1 + \frac{2}{x}}$

16. $a - b - \frac{a^2 + b^2}{a + b}$

17. $\frac{m^2 + 6m + 9}{m - 3} - (m - 3)$

18. $x - \frac{x^2 + 3xy}{x - 2y} + 3y$

19. $\dfrac{\frac{a}{1 - a} + \frac{1 + a}{a}}{\frac{1 - a}{a} + \frac{a}{1 + a}}$

1-13. Additional Exercises for Sections 1-1 through 1-7.

Exercises 1-1a':

1. Form the converse of each of the following statements.

   (a) If $2x + 1 = y$, then x is less than y ,

   (b) The sum of two numbers is even if they are each odd numbers.

   (c) $xy = 0$ only if $x = 0$.

   (d) If $a + c = b + c$, then $a = b$.

   (e) If the sum of two numbers is a multiple of 10 , then it is an even number.

   (f) $\sqrt{a^2 + b^2} = a + b$ if $(a + b)^2 = a^2 + b^2$.

   (g) $3x + 2 = 8$ only if $x = 2$.

   (h) If $a(b + c) = ab$, then $c = 0$.

   (i) If $2xy + 3 = 1$, then xy is negative.

   (j) $(a - b) - c = a - (b - c)$ if $c = 0$.

2. Rephrase each of the following in the form of "If A , then B" ; and if B , then A ."

   (a) $3x - 2 = 10$ if and only if $x = 4$.

   (b) $y = z$ if and only if $y + x = z + x$.

   (c) m is less than n if and only if $m - a$ is less than $n - a$.

   (d) $abc = 0$ if and only if $c = 0$.

   (e) $r + s = 0$ if and only if $r = -s$.

   (f) $p(r + s) = ps$ if and only if $r = 0$.

   (g) x is negative if and only if $-x$ is positive.

   (h) $a = b$ if and only if $(a - b)(a + b) = 0$.

   (i) $x + (y \cdot z) = (x + y) \cdot (x + z)$ if and only if $x = 0$.

Exercises 1-2b':

1. Which of the natural number properties is illustrated by each of the following statements? All letters represent arbitrary natural numbers.
   (a) If $x + 2 = 6$, then $x = 4$.
   (b) $(x + y)(x - y) + (x + y)^2 = (x + y) \cdot 2x$.
   (c) $2(3a) = 6a$.
   (d) $(x + 2y) \cdot x = x(x + 2y)$.
   (e) $5 + (4 + p) = 9 + p$.
   (f) $2x + (x + 3) = 3x + 3$.
   (g) $a + 2b = (a + 2b) \cdot 1$.
   (h) If $3x = 6$, then $x = 2$.
   (i) $w + 3(z + 1) = 3(z + 1) + w$.
   (j) $(a + b) \cdot c + (a + b) \cdot d = (a + b)(c + d)$.

2. Prove the following statements true for all natural numbers.
   (a) $x(y + z) = zx + yx$.
   (b) $(x + y) + z = y + (z + x)$.
   (c) $(x + y)(u + v) = y(v + u) + x(v + u)$.
   (d) $xy + y = y(1 + x)$.
   (e) $2[x + (y + 3)] = 2y + 2(x + 3)$.

Exercises 1-2c':

1. Using the natural number properties, remove all parentheses from the following products and list the properties used.
   (a) $(x + 1)(3x + 2)$.
   (b) $(2x + 1)(x + 2)$.
   (c) $2x(x + y + 3)$.
   (d) $3x(2x + y + 4)$.
   (e) $(x + 2)^2$.
   (f) $2(x + 1)^2$.
   (g) $15(2x)(3y)$.
   (h) $3x(2y)w$.
   (i) $(x + 1)(x + y + 2)$.
   (j) $(x + y + z)^2$.

2. Using natural number properties, simplify the following to a single term.

   (a) $6x + 3xy$.

   (b) $4yz + 2z$.

   (c) $3(m + 2n) + 4(m + 2n)$.

   (d) $2(3x + 1) + 5(3x + 1)$.

   (e) $a(x + y) + a(x + y)$.

   (f) $x(a + 2b) + x(a + 2b)$.

   (g) $y + 3xy$.

   (h) $5pq + p$.

   (i) $ab + ac + ad$.

   (j) $ab + ac + bd + cd$.

3. Prove the following statements for all natural numbers

   (a) $(x + y) + (w + z) = (x + y + w) + z$.

   (b) $xy + xz + yw + wz = (x + w)(y + z)$.

   (c) $(xy)(uv) = xyuv$.

   (d) $(a + b)(x + y + z) = x(a + b) + y(a + b) + z(a + b)$.

   (e) $x^2 + 2xy + y^2 = (x + y)^2$.

Exercises 1-2d':

Find natural number solutions for the following equations and list the E,A,M,D,C natural number properties used.

1. $x + 3 = 4$.
2. $y + 5 = 12$.
3. $2a = 16$.
4. $7z = 21$.
5. $3x + 6 = 18$.
6. $3 + 8z = 27$.
7. $2a + 5 = a + 8$.
8. $3p + 9 = p + 23$.
9. $4w + 5 = 6 + 5w$.
10. $3x + 15 = 6 + 5x$.

Exercises 1-3a':

1. List the members of the set of natural numbers such that $x < 4$.
2. List the members of the set of natural numbers such that $x < 2$.
3. Form an equation using natural numbers and having the same meaning as $2 < 3$.

4. Form an equation using natural numbers and having the same meaning as $5 > 1$.

5. Using the symbol " $<$ " , write true statements using the following pairs of natural numbers.

   (a) 3 and 4 .
   (b) 7 and 12 .
   (c) x and $2x$.
   (d) a and $(a + 2)$.
   (e) $(3x + 1)$ and $(2x + 4)$.
   (f) $(4m + 3)$ and $(5m + 1)$.
   (g) x and y , where $x = a + 1$ and $a = y + 2$.

6. Rewrite the following statements using $x \neq y$, $x \leq y$, or $x < y < z$.

   (a) x is less than y or y is less than x .
   (b) x is greater than y or y is greater than x .
   (c) x is less than 9 and 9 is less than y .
   (d) x is less than 5 , and 5 is less than y or 5 is equal to y .
   (e) x is less than 2 or x is equal to 2 .
   (f) 1 is less than x and x is less than 3 .
   (g) 2 is less than 3 or 2 is equal to 3 , and 3 is less than 5 .
   (h) 4 is greater than x or 4 is equal to x .
   (i) 2 is less than x or 2 is equal to x , and x is less than 5 or x is equal to 5 .
   (j) x is less than or equal to y , but y is less than or equal to z .

Exercises 1-3b':

1. Solve the following for natural numbers.

   (a) $3x < 9$.
   (b) $24 > 6y$.
   (c) $3m + 2 < 23$.
   (d) $16 > 5w + 1$.
   (e) $7x + 3 \leq 17$.
   (f) $23 \geq 6c + 5$.
   (g) $5z + 1 < 2z + 7$.
   (h) $4y + 3 \geq 6y + 1$.
   (i) $5 < 2x + 1 < 7$.
   (j) $26 > 7x + 5 > 19$.

2. Prove the following for natural numbers.

   (a) If $x < y$, then $x < y + z$.
   (b) If $x(y + z) = wz$, then $x < w$.
   (c) If $x(y + z + w) = a$, then $x(z + w) < a$.
   (d) If $x > y$ and $w > z$, then $x + w > y + z$.
   (e) If $x = a + b$ and $a < y$, then $2a < x + y$

__Exercises__ __1-4a':__

1. Find additive inverses for the following integers.

   (a) $6 - 2$.
   (b) $4 - 9$.
   (c) $x - 2x$.
   (d) $-x + 1$.
   (e) $0 - y$.
   (f) $-(-x)$.

2. Which of the theorems or definitions for I are illustrated by each of the following? All letters represent arbitrary integers.

   (a) If $a + x = b$ and $a + y = b$, then $x = y$.
   (b) $2m + 0 = 2m$.
   (c) If $m + n = 0$, then $n = -m$.
   (d) If $p + (-p) = 0$, then $-p = 0 - p$.
   (e) $(x + 2y) + [-(x + 2y)] = 0$.
   (f) If $s = -a$, then $a = -s$.
   (g) $0 - (-4) = 4$.
   (h) If $p \neq 0$ and p is not a natural number, then $-p$ is a natural number.

3. Prove that $(a + b) - c = a + (b - c)$. (Hint: Let $x = (a + b) - c$ and $y = b - c$ and show that $c + x = a + (c + y)$.)

4. Prove that $a-(b - c) = (a - b) + c$. (Hint: Let $x = b - c$, $y = a - b$, and show that $y + c = a - x$.)

5. Prove that $a -(b + c) = (a - b) - c$.

Exercises 1-4b':

1. Perform the indicated operations and list the properties or theorems used. All letters represent arbitrary integers.
   (a) $(x + y)(-1)$.
   (b) $(-3)(-x)$.
   (c) $6 - (-2)$.
   (d) $(-3) \cdot 4$
   (e) $5(0)(a - b)$.
   (f) $(-x) + (-2)$.
   (g) $-2(3)(4)$.
   (h) $(-8) + 12$.
   (i) $(-4) - (-7)$.
   (j) $(-5) - (-9)$.

2. Solve each of the following equations in I and state the E,A,M,D,C properties used.
   (a) $4x - 2 = 8$.
   (b) $6m + 1 = 13$.
   (c) $5y - 3 = 2y + 6$.
   (d) $3p + 7 = p + 9$.
   (e) $4x - 2(x + 1) = 6$.
   (f) $7y + 3(2y + 3) = 17$.
   (g) $4(a + 7) + 3 = 6 + 3(2a + 5)$.
   (h) $5 - 2(3x + 4) = 3(x + 2) - 18$.
   (i) $3(y - 1) + 2 = 6 - 2(y + 3)$.
   (j) $13 - (3w - 4) = 1 - 2(1 - 3w)$.

3. Prove each of the following statements for all integers.
   (a) $a - (b - c) = a - b + c$.
   (b) $a(b - c) = ab - ac$.
   (c) $(a - b)(a + b) = a^2 - b^2$.
   (d) $(a - b)^2 = a^2 - 2ab + b^2$.
   (e) $(a - b)(a^2 + ab + b^2) = a^3 - b^3$.

Exercises 1-5a':

1. Use the symbols " $<$ " and " $\leq$ " to form true statements of order for the following integer pairs:
   (a) 4 and -6 .
   (b) -2 and -3 .
   (c) -5 and 2 .
   (d) x and $-x$ if $x < 0$.
   (e) y and 1 if $0 < y$.
   (f) $2w$ and $3w$ if $w \leq 0$.
   (g) $-3z$ and z if $z < 0$.
   (h) $(y - 1)$ and $(y + 1)$ if $y \leq 0$.
   (i) $2x$ and $-2x$ if $0 \leq x$.
   (j) $(2p + 1)$ and $(2p - 1)$ if $0 \leq p$.

2. Prove for arbitrary integers x, y, w, and z

(a) If $0 < x$ and $0 < y$, then $x < x + y$.
(b) If $x < y$, then $2x < x + y$.
(c) If $x < y$, then $x - y < 0$.
(d) If $x < y$ and $z < w$, then $x - w < y - z$.
(e) If $0 < x < y$, then $y - x < x + y$.

Exercises 1-5b':

1. Solve the following inequalities.

(a) $2x - 3 < 11$, x in N.
(b) $4 - 3y > -21$, y in N.
(c) $5z - 4 < 2z + 5$, z in N.
(d) $6m + 10 > 8m + 6$, m in N.
(e) $2(c + 1) - 3 \leq 8 - c$, c in N.
(f) $4(1 - 2c) + 7 \geq -3 - 3(c + 2)$, c in I.
(g) $-1 < 2x + 1 < 1$, x in I.
(h) $-5 \leq 3x + 1 < 10$, x in I.
(i) $-1 < 3 - 2y < 1$, y in I.
(j) $2w - 1 \leq 3w + 1 < 4w + 3$, w in I.

2. Find solutions for each of the following where all letters represent integers.

(a) $|x| = 3$.
(b) $|y| + 4 = 0$.
(c) $|z + 3| < 2$.
(d) $|m - 5| < 6$.
(e) $|2a + 1| = 7$.
(f) $-|1 - 2x| = 5$.
(g) $4 + 3|2x - 1| \leq 13$.
(h) $13 - |6 - 3x| = 4$.
(i) $18 - 2|y + 3| \geq 12$.
(j) $4 - |2x - 1| \geq -1$.

Exercises 1-6a':

1. Solve each of the following equations if all letters represent arbitrary integers.

   (a) $3x - 1 = 6$.
   (b) $5y + 3 = 5$.
   (c) $2(m + 1) = 3 - (m + 2)$.
   (d) $4 - 3(w - 2) = 5w + 1$.
   (e) $5x + 3(1 - x) = 6$.
   (f) $3x + a = b$.
   (g) $ay + 2 = 3b$, $a \neq 0$
   (h) $2(x - a) + 1 = 3a + 4$.
   (i) $a - bx = c$, $b \neq 0$
   (j) $ax + b(c - x) = d$, $a \neq b$

2. For what values of K will each of the following pairs of rational numbers be equal?

   (a) $\frac{4}{5}$, $\frac{K}{20}$.
   (b) $\frac{2}{3}$, $\frac{8}{K}$, $K \neq 0$.
   (c) $\frac{7}{K}$, $\frac{21}{105}$, $K \neq 0$.
   (d) $\frac{K - 1}{6}$, $\frac{5}{2}$.
   (e) $\frac{2}{7}$, $\frac{6}{3K + 1}$, $K \neq -\frac{1}{3}$.
   (f) $\frac{2}{K + 1}$, $\frac{3}{2K - 1}$, $K \neq -1$, $K \neq \frac{1}{2}$.

Exercises 1-6b':

1. Find each of the following sums; all letters represent arbitrary integers.

   (a) $\frac{3}{4} + \frac{2}{3}$.
   (b) $\frac{x}{2} + \frac{y}{3}$.
   (c) $\frac{5}{7} + \frac{a}{b}$, $b \neq 0$
   (d) $\frac{3}{x} + \frac{5}{y}$, $x, y \neq 0$
   (e) $a + \frac{2}{3}$.
   (f) $\frac{a}{b} + \frac{c}{b}$, $b \neq 0$
   (g) $\frac{2}{a - b} + \frac{c}{d}$, $a \neq b$, $d \neq 0$
   (h) $\frac{a + 1}{3} + \frac{a - 1}{5}$.
   (i) $\frac{a + 2}{b} + (a - 2)$, $b \neq 0$
   (j) $\frac{a}{a - b} + \frac{a}{b - a}$, $a \neq b$

2. Find each of the following products; all letters represent arbitrary integers.

   (a) $\frac{3}{4} \cdot \frac{5}{7}$.

   (b) $\frac{x}{2} \cdot \frac{y}{3}$.

   (c) $\frac{5}{7} \cdot \frac{a}{b}$, $b \neq 0$.

   (d) $a \cdot \frac{2}{3}$

   (e) $\frac{a}{b} \cdot \frac{c}{b}$. $b \neq 0$.

   (f) $\frac{z}{a - b} \cdot \frac{c}{d}$, $a \neq b$, $d \neq 0$.

   (g) $\frac{a + 1}{3} \cdot \frac{a - 1}{5}$.

   (h) $\frac{a + 2}{3} \cdot (a - 1)$.

   (i) $\frac{2a + 3}{b} \cdot 0$, $b \neq 0$.

   (j) $\frac{a}{a - b} \cdot \frac{a}{b - a}$, $a \neq b$.

Exercises 1-6c':

1. Find solutions for each of the following and list the properties of the rational number system used. All letters represent arbitrary rational numbers.

   (a) $\frac{5x}{3} = 4$.

   (b) $\frac{2y}{7} + 1 = 6$.

   (c) $2w + \frac{3}{4} = 1$.

   (d) $5n - \frac{2}{3} = 4$.

   (e) $\frac{2x - 1}{3} + 1 = \frac{3}{4}$.

   (f) $\frac{2}{3}(y - 1) + \frac{1}{2} = \frac{3}{5}$.

   (g) $\frac{3(x + 5)}{4} = \frac{2(3x - 1)}{5}$.

   (h) $\frac{2y + 3}{5} - \frac{y + 1}{2} = 6y$.

   (i) $1 - \frac{2(1 - x)}{x - 1} = \frac{x + 2}{3}$.

   (j) $\frac{3}{2} - \frac{2x - 1}{3} = x$.

2. Prove each of the following where all letters represent arbitrary rational numbers except as noted.

   (a) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a + b}{b} = \frac{c + d}{d}$. $b, d \neq 0$

   (b) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$. $a, b, c, d \neq 0$

   (c) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a + b}{a} = \frac{c + d}{c}$. $a, b, c, d \neq 0$

   (d) If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a - b}{b} = \frac{c - d}{d}$. $b, d \neq 0$

Exercises 1-7a':

1. Determine the order relation between the following pairs of rational numbers.

   (a) $\frac{11}{7}, \frac{13}{8}$.   (d) $\frac{m+1}{7}, \frac{m-1}{9}$.

   (b) $\frac{2}{31}, \frac{4}{61}$.   (e) $\frac{5}{3a}, \frac{28}{17b}$; $a < b$, $a,b \neq 0$

   (c) $\frac{2x}{3}, \frac{9y}{13}$; $x < y$.

2. Write a chain of inequalities using the following:

   $\frac{1}{3}, -\frac{15}{16}, \frac{13}{7}, -\frac{26}{13}, \frac{25}{13}, -\frac{53}{27}, \frac{12}{35}$.

3. Prove $\frac{1}{a} < 0$ if and only if $a < 0$, $a \neq 0$, and a in I.

4. Prove: If $\frac{a}{b} > \frac{c}{d}$ and $\frac{c}{d} > \frac{e}{f}$, then $\frac{a}{b} > \frac{e}{f}$ when a,b,c,d,e,f in I, $b,d,f \neq 0$.

Exercises 1-7b':

1. Find five rational numbers between $\frac{5}{7}$ and $\frac{6}{7}$.

2. Write a chain of inequalities using the following:

   $|\frac{5}{6} + \frac{3}{4}|$, $|\frac{5}{6} - \frac{3}{4}|$, $|\frac{5}{6} \cdot \frac{3}{4}|$, $|\frac{3}{4}| - |\frac{5}{6}|$.

3. Prove $-|a| \leq a \leq |a|$ for a in Q.

4. Prove $|ab| = |a| \cdot |b|$ for a,b in Q.

Exercises 1-7c':

1. $6 < 3x + 2 < 10$.
2. $-4 < 2y + 7 < 4$.
3. $-2 < \frac{2w + 3}{5} < 2$.
4. $-1 < 3 - x < 1$.
5. $2 < \frac{5 - 3y}{2} < 15$.
6. $2 < 3 - 2m \leq 3$.
7. $-3 \leq 4 - 2a < 3$.
8. $-\frac{1}{2} < 3p - \frac{1}{3} < \frac{1}{2}$.
9. $-1 \leq \frac{4 - 3x}{-2} \leq 1$.
10. $-1 \leq \frac{4 - 5x}{-2} \leq 1$.

Exercises 1-7d':

1. Solve the following for x in Q.

(a) $|x + 3| < 5$.
(b) $|x + 1| < 7$.
(c) $|3y - 2| \leq 2$.
(d) $|2y - 5| \leq 5$.
(e) $|4 - m| > 6$.
(f) $|2 - p| > 4$.
(g) $\left|\frac{6 - 2x}{3}\right| \geq 2$.
(h) $\left|\frac{4 - 3x}{2}\right| \geq 1$.
(i) $1 \leq |x + 2| \leq 3$.
(j) $-1 \leq |2x - 3| \leq 1$.

2. Prove Theorems 1-7a,b for $a = 0$.

1-14. Miscellaneous Exercises.

1. In which of the number systems, N, I, Q, or R, does each of the following equations have a solution?

(a) $2x + 6 = 0$.
(b) $\frac{2}{3}x = 5$.
(c) $2 + x = 3x + 6$.
(d) $1 + x = 1$.
(e) $x^2 = 4$.
(f) $23x - 5x = 7x + 6(1 + 2x) + 1$.
(g) $3.5x + 14 = 2.1x + 60.2$.
(h) $x^2 - 2 = 0$.
(i) $6x + 7(4 + x) - 6(3 - 4x) = 0$.
(j) $\frac{x + 5}{2} - \frac{2x + 4}{3} = 1$.

2. Identify each of the following statements as being true or false.

(a) $\{-1, 0, 1\}$ is closed under addition.
(b) The set of natural numbers contains a greatest element.
(c) The set of integers contains a least element.
(d) The sum of a number and its additive inverse is zero.
(e) For each rational number $x \neq 0$ there is a rational number $\frac{1}{x}$.

(f) Every equation over the real numbers has a solution.

(g) $\sqrt{(5 - 8)^2} = 3$.

(h) $0 \leq x^2$ for x a real number.

(i) $\sqrt{2} + \sqrt{5} = \sqrt{7}$.

3. Which of the E,A,M,D,O properties of the real numbers are illustrated by the following?

   (a) $3[4(x + 2)] = 12(x + 2)$.

   (b) If $5 = y + z$ and $2y - z = 5$, then $y + z = 2y - z$.

   (c) If $m < n$, then $mn^2 < n^3$.

   (d) $p + 2 = (p + 2) \cdot 1$.

   (e) If $\frac{a}{b} + \frac{c}{b} = \frac{a + c}{b}$, then $\frac{a + c}{b} = \frac{a}{b} + \frac{c}{b}$. $(b \neq 0)$

   (f) $(x + y)(x - y) = (x - y)(x + y)$.

   (g) $(r + s) + q(r + s) = (1 + q)(r + s)$.

   (h) $\frac{c}{5} + \frac{0}{7} = \frac{c}{5}$.

   (i) If $x + 3 < y + 2$, then $x < y - 1$.

   (j) $-a + b = b - a$.

4. Determine which of the following statements are true. For those which are true, list the natural number properties involved.

   (a) $6(4 + 5) = 6 \cdot 4 + 6 \cdot 5$.

   (b) $6 + (4 \cdot 5) = (6 + 4) \cdot (6 + 5)$.

   (c) $73 + 7 = 7 + 73$.

   (d) $6(7 + 4) = 6 \cdot 7 + 4$.

   (e) $6 \cdot 19 = 19 \cdot 6$.

   (f) $5(20 \cdot 17) = (5 \cdot 20) \cdot 17$.

(g) $5 \cdot (3 \cdot 4) = (5 \cdot 3) \cdot (5 \cdot 4)$.

(h) $(5 + 7) + 6 = (7 + 5) + 6$.

(i) $(5 + 7) + 6 = 6 + (5 + 7)$.

(j) $(3 \cdot 4) + (5 \cdot 6) = (4 \cdot 3) + (5 \cdot 6)$.

(k) $(12 + 88) \cdot (100 + 10) = (12 + 88) \cdot 100 + (12 + 88) \cdot 10$.

(l) $(3 \cdot 7) + (8 \cdot 2) = (8 \cdot 2) + (3 \cdot 7)$.

5. Explain why you get the same answer whether you add a column of figures up or down.

6. Use the symbol " < " to state order relations for each of the following pairs of real numbers.

(a) $6, -3$.

(b) $-2, -5$.

(c) $-7, 0$.

(d) $8.2536, 8.2535$.

(e) $-0.1, -0.001$.

(f) $\frac{7}{5}, \frac{4}{5}$.

(g) $\frac{4}{23}, -\frac{3}{23}$.

(h) $\frac{5}{7}, \frac{6}{13}$.

(i) $-\frac{4}{15}, -\frac{5}{19}$.

(j) $\frac{5}{7}, \frac{12}{17}$.

(k) a, a^2 for $1 < a$.

(l) a, a^2 for $|a| < 1$.

(m) $a, -a$ for $a < 0$.

(n) a, a^2 for $0 < a < 1$.

(o) a, a^2 for $-1 < a < 0$.

7. Solve the inequalities as noted.

(a) $2x + 3 < 7$, x in N .

(b) $3y - 2 \leq 10$, y in N .

(c) $4p - 5 < 2p + 2$, p in N .

(d) $\frac{2m}{3} + 1 > 3m - 1$, m in N .

(e) $2 < 5x - 4 < 6$, x in I .

(f) $\frac{c}{2} - 1 < 3 - c$, c in N .

(g) $\frac{2x}{3} + 1 < \frac{5(x - 1)}{2}$, c in N .

(h) $\frac{7y - 5}{2} < \frac{5y + 4}{5}$, y in N .

(i) $1 \leq \frac{2 - 3x}{3} < 4$, x in Q .

(j) $0 \leq 2w + 5 \leq 10$, w in Q .

(k) $3 - y < \frac{y}{2} - 1 \leq 10$, y in R .

(l) $7d + 8 < 5d - 4 < 10$, d in I .

(m) $\frac{x - 3}{x} < 1$, x in R .

8. Solve the following inequalities or equations as indicated.

(a) $|x| < 4$, x in I .

(b) $|y| > 4$, y in Q .

(c) $|c| \leq -2$, c in R .

(d) $|2p| < 8$, p in N .

(e) $|3n| \geq 5$, n in I .

(f) $|\frac{x}{2}| + 1 = 4$, x in I .

(g) $|3m - 1| + 3 < 2$, m in N .

(h) $|\frac{x + 1}{2}| + x = 6$, x in Q .

(i) $|\frac{4y + 3}{2}| = y - 1$, y in Q .

(j) $4 + |\frac{2c - 1}{3}| < 6$, c in Q .

(k) $|2m + 1| < -1$, m in I .

(l) $x - |\frac{x}{2}| = 4$, x in R .

(m) $3 < |2z + 1| < 5$, z in R .

(n) $|x| \leq 3$ or $|x| \geq 5$, x in I .

9. Prove the following statements for real numbers.

   (a) $x - (y + z) = (x - y) - z$.

   (b) $(x - y) + (w - z) = (x + w) - (y + z)$.

   (c) $0 < x$ if and only if $-x < 0$.

   (d) $x < 0$ if and only if $0 < -x$.

10. Prove the following statements for natural numbers using only the E,A,M,D properties for natural numbers.

    (a) $x(y + z) = (z + y)x$.

    (b) $(x + y) + z = (x + z) + y$.

    (c) $(x + y)z = xz + yz$.

    (d) $x + (y + z) = (y + x) + z$.

    (e) $(x + y)(w + z) = xw + xz + yw + yz$.

Chapter 2

AN INTRODUCTION TO COORDINATE GEOMETRY IN THE PLANE

2-1. The Coordinate System.

Although you may have encountered coordinate systems before, the ideas of this section are so fundamental and so useful that we shall state them again.

Coordinate geometry, or analytic geometry, provides a means of treating geometric problems by algebra. It was first invented by a French mathematician named Rene Descartes (1596-1650) in 1637. One very great advantage of analytic geometry over synthetic geometry is that it does not depend so much on ingenuity. You will recall how very clever you needed to be to solve some of the so-called "original problems" of plane geometry. Coordinate geometry enables one to attack such problems by a straightforward method. The resulting algebra may be long and involved, but leads inevitably to the desired result. Another important use of analytic geometry is in the illumination of algebraic work. We shall see, for example, in Chapters 7 and 8 how the algebra involved in solving simultaneous equations takes on more meaning when viewed in connection with the geometric curves or surfaces which the equations represent.

You will recall that the connection between plane geometry and algebra is made by the introduction of coordinate axes in a plane. These are two perpendicular straight lines intersecting in a point O, called the origin (Fig. 2-1a). The lines are usually placed parallel to the edges of the paper so that they can be described in an obvious way as horizontal and vertical. Let us call the horizontal one the x-axis, the vertical one the y-axis and label them with letters x and y, as indicated.

Recall that on each axis we introduce a number scale, usually using the same unit on each axis, with the point O as the zero point on each scale. In statistical graphs, for example, it is often desirable to use different scales on the two axes to distort or to emphasize. But for our purposes the scales will be the same. The scales are to be so chosen that points to the right of O on the x-axis and points above O on the y-axis correspond to positive numbers. Such terms as "right" and "above" have meaning if we agree to the position of the axes described earlier.

Now comes the vital point. We establish a one-to-one correspondence between the set of all points in the plane and the set of all ordered pairs of real numbers. This means that each point P of the plane will have corresponding to it a single pair of real numbers, a first and a second (and hence ordered); and conversely that each such ordered pair of numbers will have corresponding to it just one point of the plane. How is this correspondence to be established? If P is given, project it perpendicularly first on the x-axis, second on the y-axis and read off the corresponding numbers from the scales. (The perpendicular projection of a point P on a line L is the point of intersection of L and the line through P perpendicular to L.) If the number pair is given, erect perpendiculars on the axes at the appropriate points. The point P associated with the number pair is the unique intersection of these perpendiculars. The two numbers corresponding to P are called the coordinates of P, the first its x-coordinate or abscissa, the second its y-coordinate or ordinate. We place these two coordinates in parentheses ordering them from left to right. In Fig. 2-1a the point P, for example, corresponds to the pair (-2.5,2); -2.5 is the abscissa of P and 2 is its ordinate.

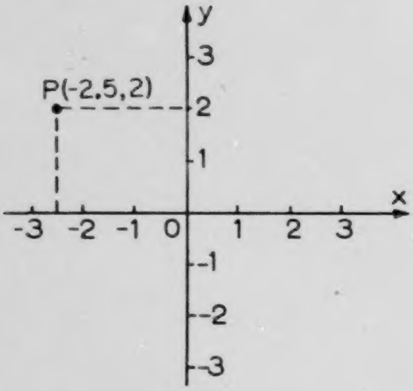

y
3
2
1
-1
-2
-3
x
-3 -2 -1 0 1 2 3
P(-2.5,2)

Fig. 2-1a.

Example 1. Plot the point P(-3,-2). What are the coordinates of its projections on the two axes?

Solution: The projection M of P on the x-axis has coordinates (-3,0); the projection N of P on the y-axis is (0,-2).

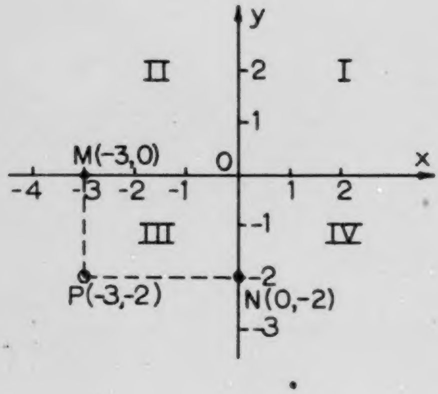
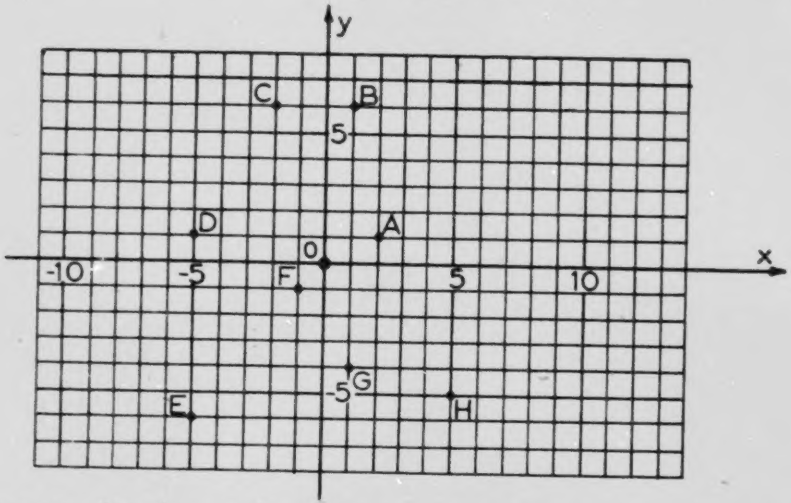

y
II
I
2
1
M(-3,0)
0
x
-4 -3 -2 -1 1 2
III
IV
-1
-2
P(-3,-2)
N(0,-2)
-3

Fig. 2-1b.

Notice that the coordinate axes divide the plane into the four regions labeled I, II, III, and IV in Fig. 2-1b. The region I is called the first quadrant, the region II the second quadrant, etc. Points on the axes are considered to be on the boundary lines and not in either of these quadrants.

Exercises 2-1

1. Locate the following ordered pairs on one set of axes: (1,5), (-3,2), (-4,-7), (5,-3), (13,2), (-6,2), (-10,-1). (Be sure to label each point by means of its coordinates.)
2. Give the coordinates of the following points.



3. Locate points (5,0) and (1,6) and connect them with a straight line. Locate points (-2,-12) and (5,9) and connect them with a straight line. What are the coordinates of the point of intersection?
4. P(4,4) lies on a circle with its center at the origin.
   (a) Draw a line from P(4,4) through the origin. Find the coordinates of a second point of the circle on this line.

(b) Draw perpendiculars from P(4,4) to both axes. Locate (i.e., give coordinates of) points on the circle other than P(4,4) which lie on the intersection of the circle with each of these perpendiculars.

(c) Can you give the coordinates of intersection of this same circle with the axes?

(d) Indicate the location of points whose number pairs satisfy:

(1) $x^2 + y^2 > 32$

(2) $x^2 + y^2 < 32$

(3) $x^2 + y^2 \geq 32$

*5. Repeat Problem 4 for the point $P(x_1,y_1)$.

6. Draw through the origin the line L which bisects the first and third quadrants.

(a) Find y for each of the following points on L: (2,y), (8,y), (-4,y), and (0,y).

(b) Write an equation in x and y which will be true for every point (x,y) on L.

7. One vertex of a square is the point A(6,6). The diagonals of this square pass through the origin.

(a) Draw the square and find the coordinates of its other vertices.

(b) Where do the sides of the square cross the coordinate axes?

(c) What is the length of its diagonals?

8. Plot the points A(6,0), B(0,6), C(0,0). What is the length of each side of triangle ABC? What is its area?

9. Draw a line segment through O(0,0) and A(6,8) extending into the third quadrant to A' chosen so that length OA = length OA'. What are the coordinates of A'? What is the length of AA'?

10. Draw the line segment connecting A(0,10) and B(12,0). Let M be the midpoint of $\overline{AB}$. Draw perpendiculars $\overline{MA'}$ and $\overline{MB'}$ to the y and x axes respectively.
    (a) What are the coordinates of A'?
    (b) What are the coordinates of B'?
    (c) What are the coordinates of M?

*11. Plot A(-3,1) ~~and~~ B(5,3). Draw segment AB. Letter the projection of B on the x-axis as C, the projection of A on the x-axis as D.
    (a) Give the coordinates of D.
    (b) Give the coordinates of C.
    (c) Give the length of segment DC.
    (d) Give the coordinates of the projection of the midpoint M of segment AB on the x-axis.
    (e) Give the coordinates of M.

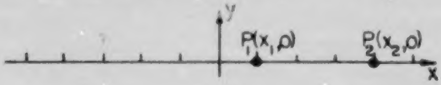
2-2. The Distance Between Two Points.

In this section we derive a fundamental formula which is useful in formulating and solving many problems in analytic geometry, the formula for the distance between two points.

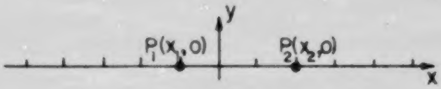
Suppose the two points are called P_1 and P_2. Let us denote the coordinates of P_1 by (x_1,y_1) and the coordinates of P_2 by (x_2,y_2). This notation is extremely useful in analytic geometry. The use of letters with subscripts for the coordinates implies that the points may represent any pair of points in the plane, but at the same time allows us to fix our attention on a particular pair for this discussion. Let us also denote the distance *between* P_1 and P_2 by $d(P_1,P_2)$. Unless otherwise stated, distance will always be non-negative; that is $d(P_1,P_2) \geq 0$.

If the two points happen to be on the x-axis, the problem is rather easy. In this case the coordinates of P_1 and P_2 become $(x_1,0)$ and $(x_2,0)$, respectively. Suppose first that P_2 is to the *right* of P_1; that is $x_2 > x_1$. Then $d(P_1,P_2) = x_2 - x_1$.

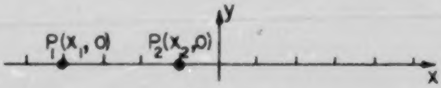
The following diagram illustrates the three possible cases.



$d(P_1,P_2) = x_2-x_1 = 4-1 = 3.$

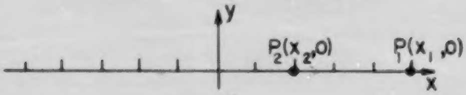


$d(P_1,P_2) = x_2-x_1 = 2-(-1) = 3.$

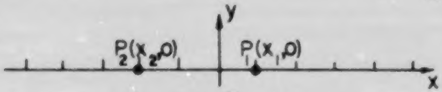


$d(P_1,P_2) = x_2-x_1 = -1-(-4) = 3.$

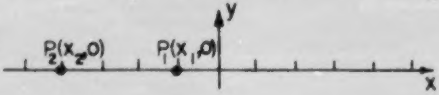
If P_2 is to the left of P_1; that is $x_2 < x_1$, then



$d(P_1,P_2) = x_1-x_2 = 5-2 = 3.$



$d(P_1,P_2) = x_1-x_2 = 1-(-2) = 3.$



$d(P_1,P_2) = x_1-x_2 = -1-(-4) = 3.$

In either case $d(P_1,P_2)$ can be written $|x_2 - x_1|$.

Similarly if P_1 and P_2 had been on the y-axis

$$d(P_1,P_2) = |y_2 - y_1|.$$

We are now ready to return to the original problem, in which P_1 and P_2 are any two points in the plane. To find $d(P_1,P_2)$ we use the Pythagorean Theorem which asserts that in a right triangle, the square of the hypotenuse is equal to the sum of the squares of the legs. First we construct a right triangle having P_1P_2 as hypotenuse as in Fig. 2-2a. R is the point of intersection of the line through P_2 parallel to the y-axis and the line through P_1 parallel to the x-axis. Its abscissa then is the same as the abscissa of N, the projection of P_2 on the x-axis, namely x_2. Its ordinate is the same as S, the projection of P_1 on the y-axis, namely y_1. Its coordinates then are (x_2,y_1).

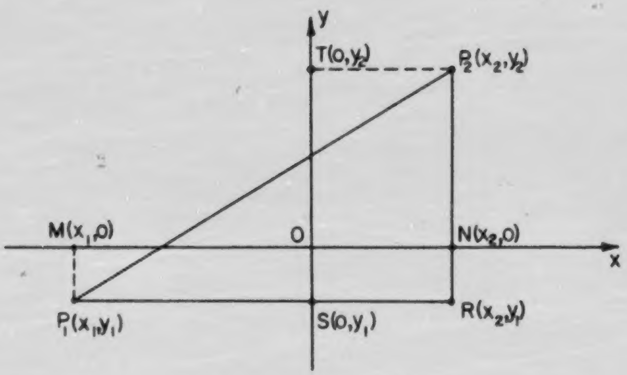


Fig. 2-2a

Then $d(P_1,R) = d(M,N) = |x_2 - x_1|$

and $d(R,P_2) = d(S,T) = |y_2 - y_1|$.

Since P_1R and RP_2 are the legs of the right triangle P_1RP_2, the Pythagorean Theorem tells us that

$$[d(P_1,P_2)]^2 = [d(P_1,R)]^2 + [d(R,P_2)]^2.$$

Substituting $|x_2 - x_1|$ and $|y_2 - y_1|$ for $d(P_1,R)$ and $d(R,P_2)$ respectively, we have

$$[d(P_1,P_2)]^2 = |x_2 - x_1|^2 + |y_2 - y_1|^2$$

or

$$d(P_1,P_2) = \sqrt{|x_2 - x_1|^2 + |y_2 - y_1|^2},$$

since all distances are non-negative.

Since $|x_2 - x_1|^2 = (x_2 - x_1)^2$ and $|y_2 - y_1|^2 = (y_2 - y_1)^2$, we have

$$d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2},$$

and we have proved the following theorem.

Theorem 2-2a: The distance between $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ is given by

2-2a
$$\boxed{d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.}$$

Example 1: Find the distance between the points

(a) $P_1(3,-2)$, $P_2(7,-5)$

(b) $P_1(-4,7)$, $P_2(-11,7)$

Solution: (a) Take $x_1 = 3$, $x_2 = 7$, $y_1 = -2$, $y_2 = -5$.

$$d(P_1,P_2) = \sqrt{(7-3)^2 + (-5+2)^2} = 5.$$

Note that we could have taken $x_1 = 7$, $x_2 = 3$, $y_1 = -5$, and $y_2 = -2$. That is, $d(P_1,P_2) = d(P_2,P_1)$.

(b) Take $x_1 = -4$, $x_2 = -11$, $y_1 = 7$, $y_2 = 7$.

$$d(P_1,P_2) = \sqrt{(-11 + 4)^2 + (7 - 7)^2}$$
$$= \sqrt{(-7)^2} = 7.$$

Since $y_1 = y_2$, the segment P_1P_2 is parallel to the x-axis.

We may now use the distance formula to prove another useful result in coordinate geometry.

Theorem 2-2b: The coordinates of the midpoint $M(x,y)$ of the line segment joining the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ are given by the formulas:

2-2b
$$\boxed{x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2}.}$$

Proof: It is enough to show that

$$d(P_1,M) = d(M,P_2) = \tfrac{1}{2}d(P_1,P_2).$$

By the distance formula (2-2a),

$$d(P_1,M) = \sqrt{(\frac{x_1+x_2}{2} - x_1)^2 + (\frac{y_1+y_2}{2} - y_1)^2} = \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2},$$

$$d(M,P_2) = \sqrt{(\frac{x_1+x_2}{2} - x_2)^2 + (\frac{y_1+y_2}{2} - y_2)^2} = \frac{1}{2}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2},$$

$$d(P_1,P_2) = \sqrt{(x_2-x_1)^2 + (y_2 - y_1)^2}.$$

Therefore $M(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2})$ is the midpoint of P_1P_2.

Example 2: Find the midpoint of the line segment joining the points (-2,5) and (0,-7).
Solution: Substituting in the midpoint formula (2-2b), we see that the required midpoint is (-1,-1).

Exercises 2-2

1. Compute the distance between the following pairs of points:
   (a) (4,-3), (-6,2);
   (b) (6,-3), (-4,-5).
2. The end points of a diameter of a circle are A(-2,4), B(4,2). Find the coordinates of the center of the circle.
3. Find the perimeter of a triangle whose vertices are A(5,7), B(1,10) and C(-3,-8).
4. A(0,8), B(-3,2) and C(10,2) are the vertices of a triangle. Find the area.
5. Find the midpoint, M, of the line segment joining the points $P_1(3,-5)$ and $P_2(0,-8)$. Check to see if the length of the segment P_1M is equal to $\frac{1}{2}$ the length of the segment P_1P_2.
6. The vertices of a quadrilateral are P(4,-3), Q(7,10), R(-8,2), and S(-1,-5). Find the length of the diagonals.

7. Plot the points A(2,3), B(-1,-1) and C(3,-4). Prove that triangle ABC is isosceles.
8. A circle whose center is at (4,-3) passes through point (9,9). Find the length of the radius. Does the circle also pass through (0,0)?
9. A line segment has a midpoint of M(3,-5) and one end is at A(2,-4). What are the coordinates of B, the other end of the segment?
10. A(-1,0) and B(-1,5) are the vertices of the base of an isosceles triangle. What are the coordinates of the third vertex C? Explain.
11. Develop a formula for the length of a line segment joining $P_1(x_1,y_1)$ and the origin.
12. Plot the points A(1,3), B(5,-1), and C(3,-3). Draw segments AB, BC, and AC. What are the coordinates of the midpoints M, N, P of these segments respectively? Find the perimeter of the triangle formed by connecting the points M, N, P. Compare the perimeter of $\triangle MNP$ with that of $\triangle ABC$.
13. Show that the points A(-4,-6), B(1,0) and C(11,12) lie on a straight line.
14. Determine the coordinates of the midpoint of the line segment joining the points $P_1(x_1,y_1)$ with the point $P_2(2x_1,2y_1)$. Find $d(P_1,P_2)$.
15. A quadrilateral has as its vertices A(2,1), B(12,3), C(6,9) and D(4,7). M, N, O, P are the midpoints of its sides $\overline{AB}$, $\overline{BC}$, $\overline{CD}$, and $\overline{DA}$ respectively.
    - (a) Plot the points.
    - (b) Find the perimeter of the quadrilateral MNOP.
    - (c) Prove that the quadrilateral MNOP is a parallelogram.

*16. A square whose sides are parallel to the coordinate axes and one vertex is (a,b) and the length of a side is c. What are the other vertices? Also, find the coordinates of the mid-points of each side of the square.

*17. Show that the points A(1,1+b), B(3,3+b), and C(6,6+b) are collinear.

(NOTE: Other problems applicable to this section may be selected from the problem-set at the end of this chapter.)

2-3. The Slope of a Line.

In plane geometry we assumed that every pair of distinct points determines a line. However a line may also be determined by one point and the direction of the line. In coordinate geometry it is useful to give the direction of a line in terms of the coordinates of any two distinct points on the line. For this purpose we define the slope, m, of the line determined by $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ to be

2-3a
$$\boxed{m = \frac{y_2 - y_1}{x_2 - x_1} \qquad (x_1 \neq x_2)}$$

However the slope of a given line, L, does not depend on the particular pair of points P_1 and P_2 used to determine the line. For, suppose P_3 and P_4 are any other two points on L. Then we construct lines through P_1 and P_2 || to the x and y axes respectively meeting in R; similarly lines through P_3 and P_4 meeting in S. See Figure 2-3a.

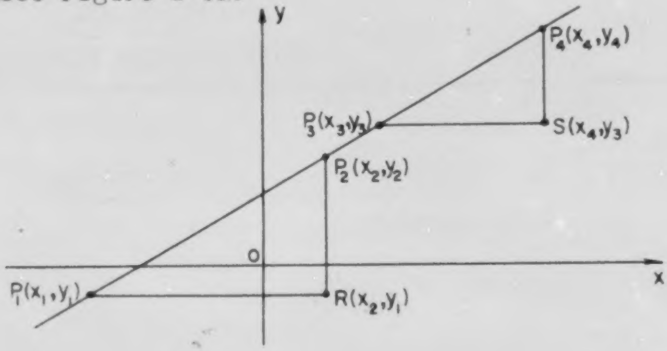


Fig. 2-3a

Triangles P_1RP_2 and P_3SP_4 are similar. Why? Therefore the corresponding sides are in proportion.

$$\frac{RP_2}{P_1R} = \frac{SP_4}{P_3S}$$

But this means $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_4 - y_3}{x_4 - x_3}$, which shows that the slope of a line is independent of the points used to determine the line.

If we consider the absolute value of m, we see that it is the quotient of $|y_2 - y_1|$ and $|x_2 - x_1|$. But from Figure 2-3a

$$|y_2 - y_1| \text{ is } d(R,P_2) \text{ and } |x_2 - x_1| \text{ is } d(P_1,R).$$

Hence the absolute value of m measures the magnitude of the steepness of the line segment P_1P_2. If we drop the absolute value symbol, the resulting quotient, m, may be positive or negative.

The sign is an important feature of the slope, for it enables us to tell whether a line rises or falls as we proceed from left to right. Let us examine the various possibilities. If the numerator and the denominator are both positive $(y_2 > y_1, x_2 > x_1)$ then P_2 is above and to the right of P_1; if both are negative $(y_2 < y_1, x_2 < x_1)$ then P_2 is below and to the left of P_1. In either case $m > 0$ and the line rises to the right. (See Figure 2-3b.)

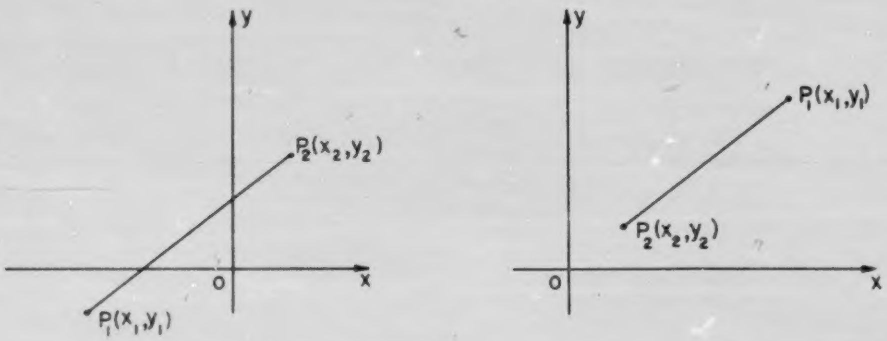


Fig. 2-3b

A similar discussion holds for $m < 0$. The line is horizontal if and only if $y_2 = y_1$, and in this case the slope $m = 0$. The line is vertical if and only if $x_2 = x_1$, in which case m is undefined.

We may summarize the preceding results as follows:

If $m > 0$, the line rises to the right.
If $m < 0$, the line falls to the right.
If $m = 0$, the line is horizontal.
If m is undefined, the line is vertical.

Example 1: Draw a line segment P_1P_2 through $P_1(2.6,-3)$ and having slope,

(a) $m = \frac{2}{3}$

(b) $m = -\frac{4}{3}$

Solution:

(a) Plot the point P_1. Starting at P_1 go three units to the right and then up 2 to reach a second point P. Note that P_2 is not uniquely determined. For, the slope may also be written $\frac{-2}{-3}$ and we could have gone 3 units to the left and down 2 units, thus arriving at a point satisfying the problem. In either case the line rises as we proceed from left to right.

(b) Go three units to the right and up (-4), that is, down 4 units. Note that the slope is negative and that the line falls as we proceed from left to right. As in the above, $-\frac{4}{3}$ may be written $\frac{-4}{3}$ or $\frac{4}{-3}$, and P_2 is again not uniquely determined.

We now use the definition of slope and the distance formula to establish two useful facts about parallel and perpendicular lines.

Theorem 2-3a: Two non-vertical lines are parallel or the same if and only if they have the same slope.

This theorem necessarily leaves out of consideration all vertical lines. But, of course, any two of them are parallel.

Theorem 2-3b. Two lines neither of which is vertical are perpendicular if and only if the product of their slopes is -1.

The following proofs of these theorems may be used as review. Even if you have seen proofs in earlier work (by similar triangles), you may enjoy reading the following alternative forms.

Proof of Theorem 2-3a: Let L_1 and L_2 be two non-vertical lines. If they coincide there is nothing to prove. Both are horizontal if and only if they are parallel to each other and to the x-axis, and hence have the same slope, namely, zero. Thus, the theorem is proved in this special case.

Assume now that neither line is horizontal. Choose any two distinct points $P_1(x_1,y_1)$, $P_2(x_2,y_2)$ on L_1, Figure 2-3b.

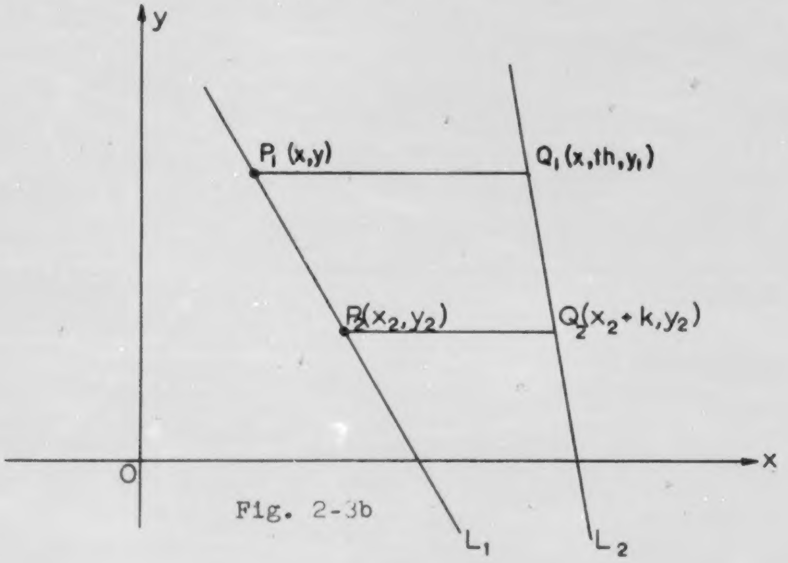


Fig. 2-3b

Draw horizontal lines through P_1 and P_2 intersecting L_2 in $Q_1(x_1 + h, y_1)$ and $Q_2(x_2 + k, y_2)$, respectively. Now the lines L_1 and L_2 are parallel if and only if $d(P_1,Q_1) = d(P_2,Q_2)$. But $d(P_1,Q_1) = d(P_2,Q_2)$ if and only if $h = k$. By the slope formula 2-3a the slopes of P_1P_2 and Q_1Q_2 are

$$\frac{y_2-y_1}{x_2-x_1} \text{ and } \frac{y_2-y_1}{x_2+k-x_1-h},$$

respectively. These two numbers are equal if and only if $h = k$. Therefore, it follows that L_1 and L_2 are parallel if and only if they have the same slopes.

Let us now turn to the proof of Theorem 2-3b. Two lines, neither of which is vertical, are perpendicular if and only if the product of their slopes is -1. In the proof of Theorem 2-3b we shall need the full statement of the Pythagorean Theorem. Although it may not have been emphasized to you, the Pythagorean Theorem works both ways. Its full statement is: The sum of the squares of two sides of a triangle is equal to the square of a third side if and only if the triangle is a right triangle.

Proof of Theorem 2-3b: Suppose we are given two non-vertical lines L_1 and L_2 with slopes m_1 and m_2, respectively. Either these lines intersect or are parallel to each other. If the latter is the case, they are certainly not perpendicular and, by Theorem 2-3a their slopes m_1 and m_2 are equal so that m_1m_2 cannot be equal to -1. Thus we need only consider the case in which L_1 and L_2 intersect. Draw lines L_1' and L_2' (if necessary) parallel to L_1 and L_2 and such that L_1' and L_2' intersect at the origin. See Figure 2-3c. By Theorem 2-3a the slopes of L_1' and L_2' are then m_1 and m_2, respectively.

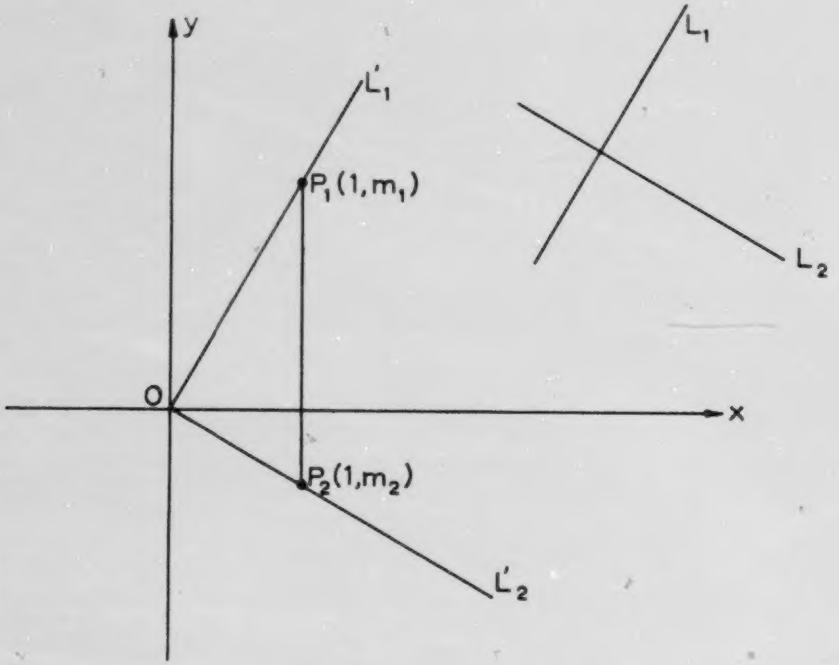


Fig. 2-3c

Consider the points $P_1(1,m_1)$ and $P_2(1,m_2)$. By Formula 2-3a the slope of OP_1 is m_1 and that of OP_2 is m_2. That is, P_1 and P_2 are on L_1 and L_2, respectively. Hence, the triangle P_1OP_2 is a right triangle with right angle at O if and only if

$$[d(P_1,P_2)]^2 = [d(O,P_1)]^2 + [d(O,P_2)]^2 \text{ (by Formula 2-2a.)}$$

$$[d(O,P_1)]^2 = (1 - 0)^2 + (m_1 - 0)^2 = 1 + m_1{}^2$$

$$[d(O,P_2)]^2 = (1 - 0)^2 + (m_2 - 0)^2 = 1 + m_2{}^2$$

$$[d(P_1,P_2)]^2 = (m_2 - m_1)^2 + (1 - 1)^2 = m_2{}^2 - 2m_1m_2 + m_1{}^2.$$

Hence $[d(P_1,P_2)]^2 = [d(0,P_1)]^2 + [d(0,P_2)]^2$ if and only if

$$m_2{}^2 - 2m_1m_2 + m_1{}^2 = 1 + m_1{}^2 + 1 + m_2{}^2$$

$$-2m_1m_2 = 2$$

$$\boxed{m_1m_2 = -1} \qquad \text{2-3b}$$

Hence, by the Pythagorean Theorem, OP_1 must be perpendicular to OP_2 and, therefore, the lines L_1 and L_2 are perpendicular. This proves Theorem 2-3b.

Example 2: Given $P_1(1,0)$, $P_2(4,4)$, $P_3(5,-3)$, $P_4(8,1)$. Show that P_1P_2 is parallel to P_3P_4 and perpendicular to P_1P_3.

Solution: m_1 for P_1P_2 is $\frac{4-0}{4-1} = \frac{4}{3}$

m_2 for P_3P_4 is $\frac{1-(-3)}{8-5} = \frac{4}{3}$

m_3 for P_1P_3 is $\frac{-3-0}{5-1} = -\frac{3}{4}$

$m_1 = m_2$ and $m_1m_3 = -1$.

Example 3: By the Pythagorean Theorem show that $P_1P_2P_3$ in Example 2 is a right triangle.

Solution: $[d(P_1,P_2)]^2 = (4 - 0)^2 + (4 - 1)^2 = 25$

$[d(P_2,P_3)]^2 = (-3 - 4)^2 + (5 - 4)^2 = 50$

$[d(P_3,P_1)]^2 = (-3 - 0)^2 + (5 - 1)^2 = 25$

$[d(P_2,P_3)]^2 = [d(P_1,P_2)]^2 + [d(P_3,P_1)]^2$

Example 4: Prove that the diagonals of a square are perpendicular to each other.

Solution: This is our first example of the proof of a geometric theorem by coordinate geometry. We consider a square whose sides have length a. Here a is an arbitrary positive number. We use the letter a instead of some specific number, such as 5, because we wish to prove the theorem for all squares. We now locate the axes so that two sides of the square lie along the positive axes and the vertex between these two sides is at the origin. The opposite vertex is then the point (a,a). See Figure 2-3d.

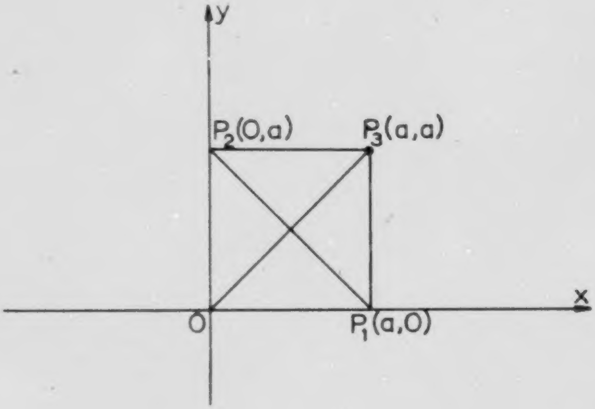


Fig. 2-3d

By Formula 2-3a the slope of P_1P_2 is $\frac{a-0}{0-a} = -1$; the slope of OP_3 is $\frac{a-0}{a-0} = 1$. The product of these slopes is -1. Therefore, the diagonals are perpendicular by Theorem 2-3b.

Exercises 2-3

1. Determine the slope of each line which passes through the following sets of points:
   (a) (10,5) and (6,8) .
   (b) (2,-2) and (4,2) .
   (c) (10,-2) and (16,1) .
   (d) (0,3) and (0,-2) .
   (e) (0,0) and (5,3) .
   (f) (-2,0) and (3,0) .

2. (a) On the same coordinate axes draw lines through $P(5,6)$, each having a slope of $\frac{1}{3}$; -3; $\frac{2}{3}$; 1; 0.
   (b) Which line is the steepest?
   (c) As the absolute value of the number for the slope increased, how do these lines compare?
   (d) What do you observe about the lines having slope of $\frac{1}{3}$ and -3?
3. (a) Plot and connect the points $(3,2)$ and $(7,-1)$; plot and connect the points $(-4,1)$ and $(0,-2)$.
   (b) Find the slope of each line
   (c) What can one say about these lines?
4. (a) Find the slopes and the lengths of the sides of a triangle having the following vertices: $A(3,2)$, $B(6,5)$, and $C(3,8)$.
   (b) What do you notice about this triangle?
   (c) Find the midpoints M_1, M_2 and M_3 of the sides of the triangle ABC.
   (d) From the data you now have in this exercise and knowledge of geometry, give the slope of $\overline{M_2M_3}$, $\overline{M_1M_3}$ and $\overline{M_1M_2}$.
5. Use the slope **formula** to show that the points $A(-4,-6)$, $B(1,0)$ and $C(11,12)$ lie on a straight line.
6. Determine b so that $A(b,5)$, $B(1,3)$ and $C(-2,1)$ are collinear.
7. The line joining $(p,2)$ and $(1,0)$ is parallel to the line joining $(2,3)$ and $(-2,1)$.
   (a) Find p.
   (b) Substituting the word "perpendicular" for the word "parallel", find p.
8. Plot the points $A(1,4)$, $B(3,2)$, $C(4,6)$, and $D(2,8)$.
   (a) Show that ABCD is a parallelogram.
   (b) Is ABCD a rectangle?

9. Plot the points $A(-3,6)$, $B(2,-3)$, $C(11,2)$ and $D(6,11)$.
   (a) Show that ABCD is a rhombus.
   (b) Show that ABCD is a square.

10. A square has its vertices located at $A(1,3)$, $B(4,3)$, $C(4,6)$ and $D(1,6)$. Show that its diagonals are perpendicular.

11. If a line L has a slope $\frac{p}{q}$, what is the slope of
    (a) a line parallel to L?
    (b) a line perpendicular to L?

12. (a) Find the slope of a line through the points $P_1(a,b)$ and $P_2(b,a)$;
    (b) Find the slope of a line perpendicular to the line through P_1P_2.

13. In the right triangle whose vertices are $A(-12,1)$, $C(9,3)$, and $B(11,-18)$, which vertex is the right angle? Explain.

14. The slope of a line through the point $(2,3)$ is $\frac{2}{3}$.
    (a) Give the coordinates of two other points which this line passes through.
    (b) Determine whether the line passes through the point $(62,23)$.

*15. A square has its vertices at $A(a,b)$, $B(a+c,b)$, $C(a+c,b+c)$, $D(a,b+c)$. Prove that the diagonals are perpendicular to each other.

*16. If a, b, and c are any real numbers, show that the points $A(a,b+c)$, $B(b,c+a)$, and $C(c,a+b)$ are collinear.

*17. A triangle has for its vertices: $A(a,b)$, $B(a+c,b)$, and $C(a+c,b+d)$.
    (a) Verify that this is a right triangle.
    (b) Determine the coordinates of the midpoint M of the hypotenuse.

2-4. Sketching Graphs of Equations and Inequalities.

We have established a one-to-one correspondence between all ordered pairs of real numbers (x,y) and all points of the plane. Suppose we wish to fix our attention on only a part of the plane and hence on a subset of all number pairs. This will impose some restriction on the numbers x and y. It may appear as a condition upon x or upon y or upon both through some relation between them. For example, every point on the y-axis has its abscissa zero, and no point off the y-axis does. Hence, the equation $x = 0$ is a restricting relation on the set of all ordered number pairs which restricts the corresponding points to lie on the y-axis. The y-axis is called the graph of the equation $x = 0$ or $x = 0$ is the equation whose graph is the y-axis. In a similar way the graph of the equation $y = 0$ is the x-axis.

Another type of restricting relation is an inequality. For example, the inequality $y > 0$ holds for those points and for only those points which lie above the x-axis; the relation $x \leq 0$ specifies the points to the left of the y-axis and those on the y-axis.

The most frequent type of restricting relation on the number pairs (x,y) is an equation between them. For example, the graph of the equation $x = y$ is evidently the line L which bisects the first and third quadrants. See Figure 2-4a.

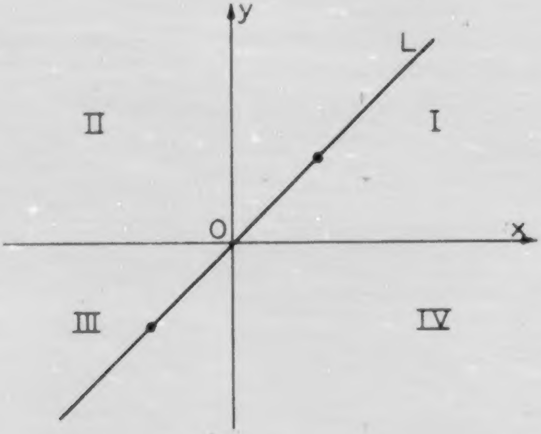


Fig. 2-4a

The set notation gives us a convenient way of describing briefly the restricting relations mentioned above. For example, the y-axis can be described by $\{(x,y) : x = 0\}$. That is, it is the set of all ordered number pairs (x,y) the first of which is zero. The line of Figure 2-4a is $\{(x,y) : x = y\}$. We now define formally what we mean by the graph of an equation or an inequality.

Definition 2-4a. *The graph of an equation or inequality in x and y is the set of all points whose coordinates satisfy the equation or inequality.*

Example 1: Sketch the graph of the equation

$$x - y - 1 = 0. \tag{2-4a}$$

Solution: Let us choose a number of values of one of the coordinates, say x, and compute the corresponding value of the other by use of the given equation. For example:

x	-2	-1	0	2	4
y	-3	-2	-1	1	3

We may now plot the corresponding set of points as sample points on the graph.

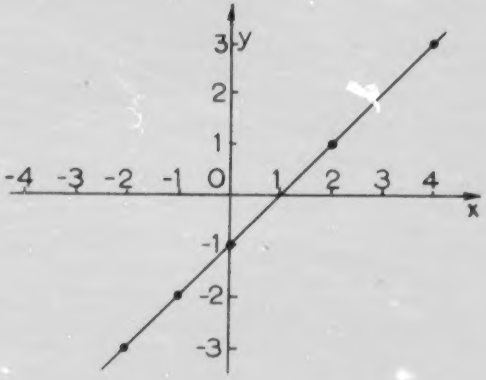


Fig. 2-4b

We sketch the graph as well as possible from the sample points. In this simple case the points seem to lie on a straight line. When we make a systematic study of specific classes of equations and their graphs in Chapter 6, we shall show that the graph of every equation of the first degree is a straight line.

Example 2: Sketch the graph of the equation

2-4b $$x^2 + y^2 = 4$$

Solution: Solve for y to obtain $y = \pm\sqrt{4 - x^2}$. A table of sample points is

x	0	1	1.5	2	-1	-2
y	± 2	± 1.7	± 1.3	0	± 1.7	0

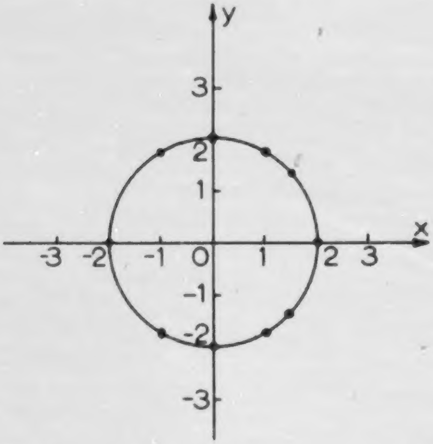


Fig. 2-4c

We have sketched in the graph as if it were a circle. By use of the distance formula (2-2a) we may check that every one of the sample points is a distance 2 from the origin. In fact, the equation $x^2 + y^2 = 4$ makes it clear that every point will have this property, and we see that the graph must be a circle.

Example 3: Sketch the graph of the equation

2-4c $$y = x^2 - 2x.$$

Solution:

x	-1	0	1	2	3
y	3	0	-1	0	3

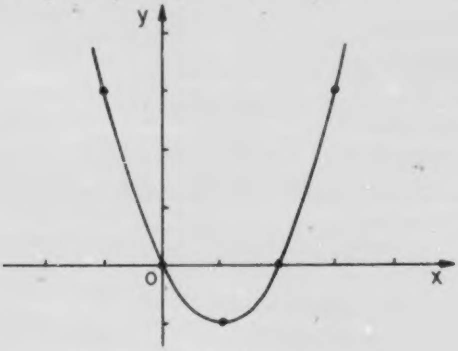


Fig. 2-4d

We have connected the sample points by a "smooth, unbroken" curve. If we wanted to check that this curve actually is the graph, we might plot additional points. However, even then we would not be sure about what happens between any two points on the curve. Better techniques than simply plotting points will be developed in the remainder of this chapter and in succeeding chapters.

Example 4: Sketch the graph of the equation

2-4d $$(x - y)y = 0.$$

Solution: We noticed in Chapter 1 that the product of two numbers is zero if and only if at least one of the numbers is zero. Hence the graph of Equation 2-4d is the combined set of points satisfying either $x - y = 0$,

or $y = 0$.

We have seen at the beginning of this section that the graph of $x - y = 0$ is the line bisecting the first and third quadrants and that the graph of $y = 0$ is the x-axis.

Therefore, the graph of the equation $(x - y)y = 0$ is the pair of intersecting lines ℓ_1 and ℓ_2 given in Figure 2-4e.

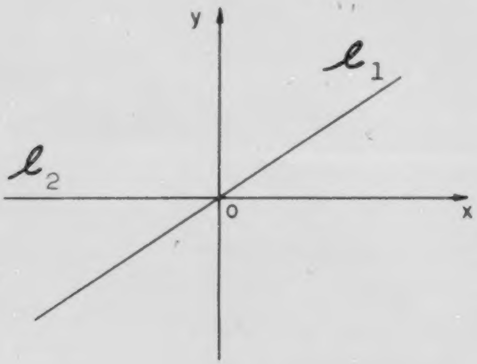


Fig. 2-4e

Example 5: What is the graph of $2x^2 + 3y^2 = -1$?
Solution: If we add 1 to both sides, we may rewrite the equation in the form

$$2x^2 + 3y^2 + 1 = 0.$$

For any real numbers x and y, x^2 and y^2 are greater than or equal to zero and accordingly $2x^2 + 3y^2 + 1$ is certainly greater than or equal to 1 and therefore greater than zero. Hence there are no points on the graph of this equation; that is, the graph is the empty set.

Example 6: Graph the inequality $0 \leq y < x$; that is, $\{(x,y) : 0 \leq y < x\}$.
Solution: Suppose we first consider a fixed value of x, say $x = 5$. What is $\{(5,y) : 0 \leq y < 5\}$? The set of all points with coordinates $(5,y)$ with no restriction on y is the straight line parallel to the y-axis passing through the point $(5,0)$. (The equation of this line is $x = 5$.) See Figure 2-4f. However we are only interested in those points $(5,y)$ on this line for which $0 \leq y < 5$. But this is just the line segment PQ with P included and Q excluded.

Thus for each fixed value of x, $\{(x,y) : 0 \leq y < x\}$ consists just precisely of the points on the line segment joining the points $(x,0)$ and (x,x), the first point included and the second excluded. Therefore the graph of the inequality is shown in Figure 2-4g.

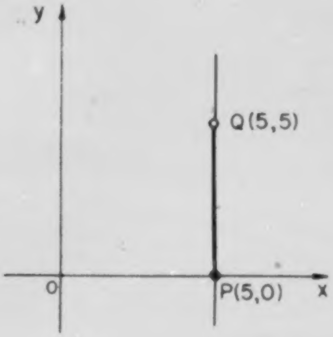


Fig. 2-4f

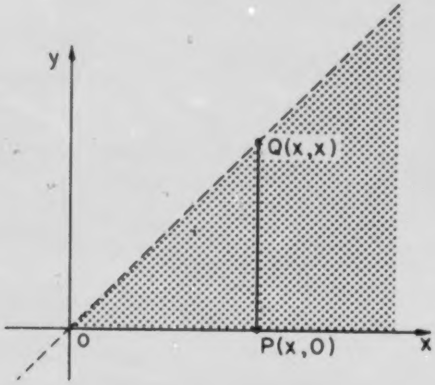


Fig. 2-4g

In Figure 2-4g the graph includes points on the x-axis for which $x > 0$, but does not include any points on the line $y = x$.

Intercepts. The abscissa of a point of a graph for which the ordinate is zero is called an x-intercept of the graph; the ordinate of a point for which the abscissa is zero is called a y-intercept. In sketching graphs it is helpful to obtain these special number pairs, if feasible. To obtain the x-intercepts, set $y = 0$ in the equation of the graph and solve for x; for the y-intercepts, set $x = 0$ and solve for y. In the equation $x^2 + y^2 = 4$ of Example 2, the x-intercepts are +2 and -2; the y-intercepts are +2 and -2. In the equation $y = x^2 - 2x$ of Example 3, the x-intercepts are 0 and 2; the y-intercept is 0.

Symmetry. In Example 2, when we solved $x^2 + y^2 = 4$ for y we obtained $y = \pm\sqrt{4 - x^2}$; that is, for every x between -2 and 2, we found two values of y which differed only in sign. A similar statement could be made if we had solved for x in terms of y. More important, we notice that if (a,b) is on the graph, so is (-a,b), and also (a,-b), and even (-a,-b). If a curve has these properties we say that it is symmetric with respect to the y-axis, the x-axis, and the origin, respectively.

We now formulate these definitions more precisely and give a few examples of the kinds of problems in which they are helpful.

Definition 2-4b: Two points are symmetric with respect to a line if the line is the perpendicular bisector of the line segment joining the points. Each point is called the reflection of the other in the line.

For example, if two points have the same abscissa and ordinates which differ only in sign, then one can be obtained from the other by a reflection in the x-axis. Thus, the points (a,b) and (a,-b) are symmetric with respect to the x-axis. Similarly, if two points have the same ordinate and abscissas which differ only in sign, then they are symmetric with respect to the y-axis; e.g., (c,d) and (-c,d). See Figure 2-4h.

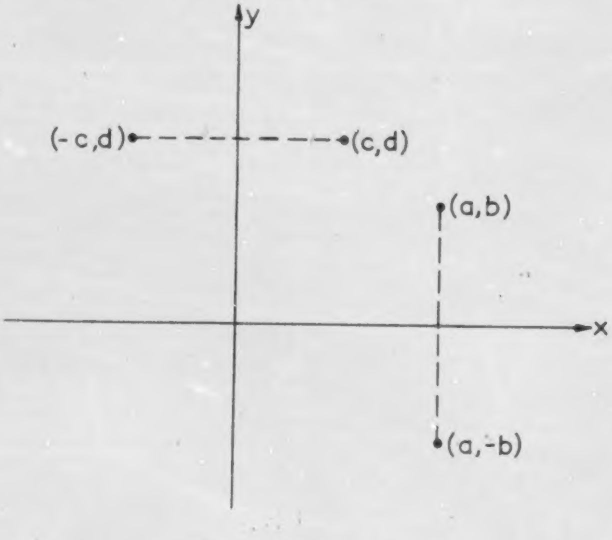


Fig. 2-4h

We shall say that a curve is symmetric with respect to a line if every point on the curve goes into another point on the curve when it is reflected in the given line. Thus the circle in Example 2 is symmetric with respect to any diameter, and the curve $y = x^2 - 2x$ in Example 3 is symmetric with respect to the line $x = 1$, which is the line parallel to the y-axis passing through the point $(1,0)$.

The following rules are worth noting. If in an equation replacing x by $-x$ results in an equivalent equation, whenever (x,y) satisfies the equation, so does $(-x,y)$. Therefore, the graph of such an equation is *symmetric with respect to the y-axis.* Similarly if an equation equivalent to the original one is obtained when y is replaced by $-y$, then the graph of the equation is *symmetric with respect to the x-axis.*

Definition 2-4c: *Two points are symmetric with respect to a point if the point is the midpoint of the line segment joining them.*

In particular the points $(-a,-b)$ and (a,b) are symmetric with respect to the origin. And we say that *a curve is symmetric with respect to the origin if every point on the curve goes into another point on the curve when it is reflected in the origin.* Accordingly a *test for symmetry with respect to the origin* is to replace x by $-x$ and y by $-y$ in the equation and if it can be made to assume its original form, the curve is symmetric with respect to the origin. For example, the equation $y = x$ becomes $-y = -x$ which can be rewritten $y = x$ by multiplying both sides by -1; the graph of this equation is therefore symmetric with respect to the origin.

Similarly $y = x^3$ becomes $-y = (-x)^3$ which is equivalent to $y = x^3$ and the curve is symmetric with respect to the origin.

On the other hand $y = x + 2$ is not symmetric with respect to the origin, since $(-x,-y)$ is not on the graph whenever (x,y) is.

It is not always simple to discover symmetries with respect to general lines or points. But whenever they are easily discovered, they should be used to simplify curve sketching. This discussion of symmetry may be summarized as follows:

The graph of an equation is symmetrical with respect to the $\begin{Bmatrix} x\text{-axis} \\ y\text{-axis} \\ \text{origin} \end{Bmatrix}$ if an equivalent equation is obtained by replacing (x,y) by $\begin{Bmatrix} (x,-y) \\ (-x,y) \\ (-x,-y) \end{Bmatrix}$

Exercises 2-4

1. Make a table of some number pairs which satisfy the following sentences. Use these to sketch the graph of each.
   (a) $2x + y - 1 = 0$
   (b) $y = x^2$
   (c) $y - x^2 = 2$
   (d) $(x - 1)y = 0$
   (e) $xy + 3x = 0$
   (f) $y = 2x^2 - x$
   (g) $y = |x|$
   (h) $x = |y - 2|$
   (i) $y > x$
   (j) $y \leq x + 3$
   (k) $x > y^2$
   (l) $y > |x|$
   (m) $x > 2$ or $y > 3$
   (n) $x > 2$ and $y > 3$
2. Plot the point $P(3,2)$.
   (a) Reflect it in the origin.
   (b) Reflect it in the x-axis.
   (c) Reflect it in the y-axis.
   (d) Reflect it in the line $y = x$.
   (e) Reflect it in the line $y = -1$.
   (f) Reflect it in the line $x = 2$.
3. Give the x and the y intercepts of the graph of
   (a) $2x - y = 6$.
   (b) $x^2 + y^2 = 1$.
   (c) $y = \frac{2}{3}x$.
   (d) $x + y^2 = 1$.
   (e) $y = x^2 - 4$.
   (f) $xy = 25$.
   (g) $y^2 + 9 = x$.
   (h) $x = |y|$.
   (i) $|x| + 15 = y$.
   (j) $x^3 + 2xy + 3y + 27 = 0$.

4. Test for symmetry with respect to the origin and the axes.

   (a) $x^2 + y^2 = 9$
   (b) $y = x^2 + 5$
   (c) $y = (x + 2)^2$
   (d) $xy = 1$
   (e) $x + y = 3$
   (f) $x^2 = y^2$
   (g) $x = y^2$
   (h) $x^2 - y^2 = 16$
   (i) $y = |x|$
   (j) $x^2 = y^3$
   (k) $3xy + 6 = 0$
   (l) $y = x^6 - x^4 + 2x^2$
   (m) $x^2y^2 - xy + 6 = 0$

5. Use the intercepts and symmetry to sketch the graph of each.

   (a) $y = 2x + 3$
   (b) $x = 2y + 3$
   (c) $y = x^2$
   (d) $y \geq x^2$
   (e) $y = -x^2$
   (f) $y = |x|$
   (g) $y < |x|$
   (h) $y = 1 - x^2$
   (i) $x = y^2$
   (j) $x = y^2 + 2$
   (k) $x^2 + y^2 = 9$
   (l) $x^2 + y^2 < 9$
   (m) $x \leq -y^2 + 4$
   (n) $(x - 2)(x - y) = 0$
   (o) $xy + x^2 = 0$
   (p) $x^2 + 4y^2 = 4$
   (q) $(x - y)(xy) = 0$
   (r) $x^2 + y = -4$
   *(s) $xy + 6 = 0$
   *(t) $9x^2 + 4y^2 > 36$
   *(u) $x^2 + 2y^2 \leq 16$
   *(v) $y = -x^3$

---

*2-5. Analytic Proofs of Geometric Theorems.

In Section 2-1 it was mentioned that coordinate geometry provides a powerful and direct means of proving geometric theorems. A simple example was given in Section 2-3. We shall give some additional ones here. All our proofs are based on the three formulas of Sections 2-2 and 2-3.

The first step in an analytic proof is the selection of the position of the coordinate axes in relation to the figure being discussed. Logically, no position of the axes is preferable. Practically, an appropriate choice of axes will simplify the coordinates of some points and reduce the algebraic work in a proof.

The next step is the assignment of coordinates to points which determine the figure. The positions of some points may be chosen arbitrarily, and these points must be assigned general coordinates; that is, letters unrelated to each other. Other points are then determined by the shape of the figure, and their coordinates must be expressed in terms of the previously chosen general coordinates.

After this has been done, the geometric relations being discussed can be expressed algebraically. The proof then proceeds algebraically.

Example 1: Prove that the median of a trapezoid is parallel to the base.

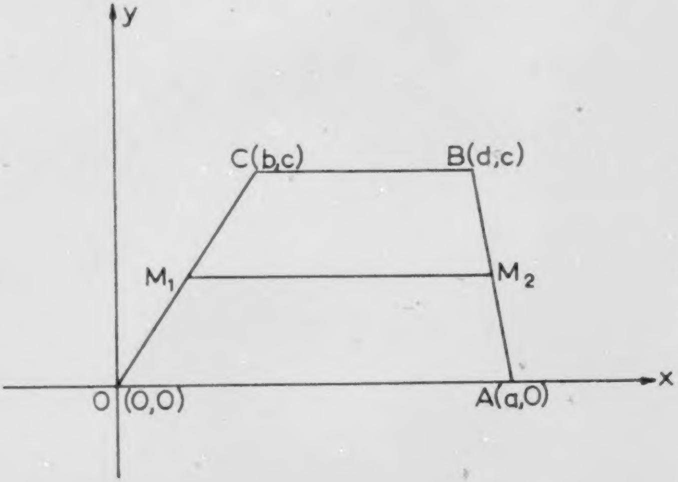


Fig. 2-5a

Solution: We introduce axes so that one of the parallel sides of the trapezoid lies on the x-axis and one vertex is at the origin, Figure 2-5a. The vertex A can lie anywhere on the x-axis so that its abscissa must be general. Accordingly A is assigned coordinates (a,0). Similarly, C can be any point in the plane, so it is assigned general coordinates (b,c). Now, however, the coordinates of B are restricted by the requirement that the side $\overline{CB}$ is parallel to the side $\overline{OA}$. This will be true if and only if the slope of CB is the same as the slope of $\overline{OA}$ by Theorem 2-3a (slopes of || lines are equal). Since the slope of OA is 0, the ordinate of B must be c. The abscissa d of B is general. By the midpoint formula 2-2b, the midpoint M_1 of $\overline{OC}$ is $(\frac{b}{2}, \frac{c}{2})$, the midpoint M_2 of $\overline{AB}$ is $(\frac{a+d}{2}, \frac{c}{2})$. By the slope formula 2-3a the slope of $\overline{M_1M_2}$ is 0. By Theorem 2-3a, $\overline{M_1M_2}$ is parallel to $\overline{OA}$, and the theorem is proved.

Example 2: Show that the diagonals of a parallelogram bisect each other.

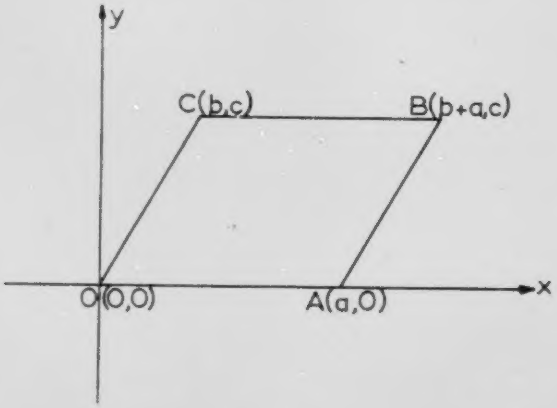


Fig. 2-5b

Solution: We place the axes as in Example 1, and again assign general coordinates (a,0) and (b,c) to A and C, Figure 2-5b. Now, however, the coordinates of B are determined by the two conditions that $\overline{CB}$ is parallel to $\overline{OA}$ and that $\overline{BA}$ is parallel to $\overline{CO}$. The first condition requires, as before, that the ordinate of B is c. If we let d denote the abscissa of B then, by Theorem 2-3a, the second condition requires that

$$\frac{c-0}{d-a} = \frac{c-0}{b-0};$$

that is, d = a + b.

Now, the midpoint of the diagonal $\overline{AC}$ is $(\frac{a+b}{2},\frac{c}{2})$ and that of $\overline{OB}$ is $(\frac{b+a}{2},\frac{c}{2})$ by the midpoint formula 2-2b. Since these midpoints coincide the theorem is proved.

Example 3: Show that the midpoint of the hypotenuse of a right triangle is equidistant from the three vertices.

Solution: Choose the axes as in Figure 2-5c.

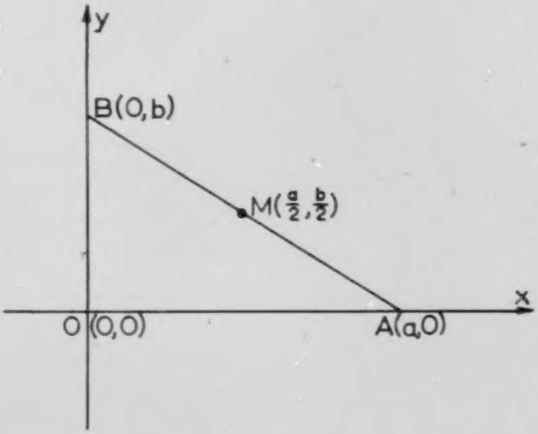


Fig. 2-5c

The midpoint M of $\overline{AB}$ has coordinates $(\frac{a}{2},\frac{b}{2})$. Hence

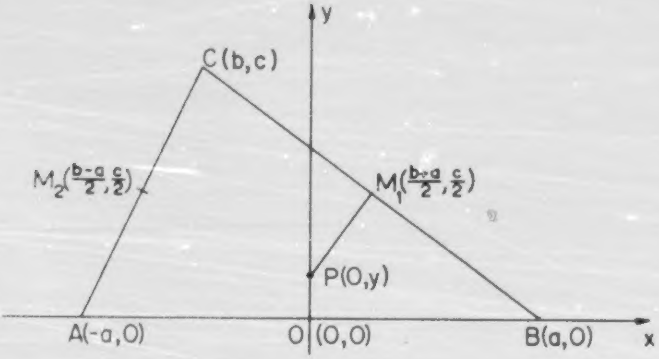
$$d(M,A) = \sqrt{(a-\frac{a}{2})^2 + (\frac{b}{2})^2} = \frac{1}{2}\sqrt{a^2 + b^2}$$

$$d(O,M) = \sqrt{(\frac{a}{2}-0)^2 + (\frac{b}{2}-0)^2} = \frac{1}{2}\sqrt{a^2 + b^2}$$

$$d(M,B) = \sqrt{(0-\frac{a}{2})^2 + (b-\frac{b}{2})^2} = \frac{1}{2}\sqrt{a^2 + b^2}.$$

Example 4: Prove that the perpendicular bisectors of the sides of a triangle meet in a point.

Solution: Choose the x-axis along one side of the triangle and the y-axis as the perpendicular bisector of this side (Figure 2-5d).



y
C(b,c)
$M_2(\frac{b-a}{2},\frac{c}{2})$
$M_1(\frac{b+a}{2},\frac{c}{2})$
P(0,y)
A(-a,0)
O (0,0)
B(a,0)
x

Fig. 2-5d

We choose general coordinates (a,0) and (b,c) for B and C. The coordinates of A are (-a,0) since O is the midpoint of $\overline{AB}$. The coordinates of the midpoints M_1, M_2 of $\overline{BC}$ and $\overline{AC}$ are determined from the midpoint formula 2-2b.

Let us find the coordinates of the point P at which the two perpendicular bisectors PO and PM_1 intersect. Since this point lies on the y-axis, it has abscissa zero. Its ordinate is an unknown number y which we have to determine. By the Formula 2-3a,

$$\text{Slope of } BC = \frac{c-0}{b-a},$$

$$\text{Slope of } PM = \frac{c-2y}{b+a}.$$

By Theorem 2-3b, concerning slopes of non-vertical perpendicular lines, the product of these slopes is -1. Thus

$$\frac{c}{b-a} \cdot \frac{c-2y}{b+a} = -1.$$

Solving this equation for y, we have

$$y = \frac{c^2-a^2+b^2}{2c}.$$

That is, the point of intersection of the two perpendicular bisectors at M_1 and at O is $(0, \frac{[c^2-a^2+b^2]}{2c})$. Now we proceed in exactly the same way to find the intersection of the two perpendicular bisectors at M_2 and at O. The data are exactly as before except that a is replaced by -a throughout. Consequently, we need not do the algebra again but have only to replace a by -a in the result. But since $a^2 = (-a)^2$, that result is unchanged and we see that the second point of intersection coincides with the first.

We emphasize the importance of assigning coordinates so that the figure determined is the most general one of its kind. There are two requirements here. The figure must have all of the properties stated in the theorem, and it must have no additional properties. Thus, in Example 1 the figure is a trapezoid, that is, a quadrilateral with two sides parallel. It would be wrong, therefore, to assign to the point B in Figure 2-5a the general coordinates (d,e), since then the points O, A, B, C would be the vertices of any quadrilateral. It would be equally wrong to assign to B the coordinates (a,c), since then the trapezoid would have a right angle at A.

Exercises 2-5

Use coordinate geometry to prove the following theorems:

1. The line joining the midpoints of two sides of a triangle is parallel to the third side and its length is one-half the length of the third side.
2. If the diagonals of a parallelogram are perpendicular, it is a rhombus.
3. If the diagonals of a quadrilateral bisect each other, it is a parallelogram.
4. The lines joining the midpoints of the sides of a rhombus form a rectangle.
5. The sum of the lengths of the perpendiculars drawn from the midpoints of two sides of a triangle to the third side equals the length of the altitude drawn to the third side.
6. The lines joining the midpoints of the sides of a triangle divide the triangle into four congruent triangles.
7. The lines joining the midpoints of the opposite sides of a quadrilateral bisect each other.
8. If one of the equal sides L of an isosceles triangle is extended by its own length through the vertex opposite the base to P, the line from P to the vertex not on L is perpendicular to the base.
9. Lines joining the midpoints of the sides of an isosceles trapezoid form a rhombus.

2-6. Sets Satisfying Geometric Conditions.

In Section 2-4 we considered the question of determining the set of points whose coordinates satisfied some restricting relation. In fact we concentrated on sets whose coordinates satisfied an equation. In this section we reverse the question and ask for an algebraic description of the set determined by some geometric condition. The machinery of analytic geometry is ideally suited for this task. We use the results of the preceding sections to write algebraic descriptions of geometric conditions.

Example 1: Describe the set of all points at a distance 1 from the origin.

Solution: Geometrically, the set of points on a circle with center at the origin and radius 1, satisfies this condition. This is a perfectly good description of the set. However, we could still describe the set algebraically by using an equation to express the given geometric condition.

Let $P(x,y)$ be any point satisfying the condition.

Then $$d(O,P) = 1.$$

Using the distance formula (2-2a)

$$\sqrt{(x - 0)^2 + (y - 0)^2} = 1$$

or $$x^2 + y^2 = 1.$$

This algebraic condition is simply a straightforward algebraic translation of the geometric condition.

Example 2: Find the set of all points which are twice as far from the origin as from the point (2,0).

Solution: In this case, we may have no idea what the geometric description of the set is. However, it is still easy to write out the algebraic description.

Suppose P(x,y) is any point of the set and let A be the point (2,0). Then

$$d(O,P) = 2d(P,A)$$

$$\sqrt{x^2 + y^2} = 2\sqrt{(x - 2)^2 + (y - 0)^2}$$

or $$x^2 + y^2 = 4[(x - 2)^2 + (y - 0)^2].$$

Simplifying we get $3x^2 - 16x + 16 + 3y^2 = 0$.
In Chapter 6 we shall show that this set is actually a circle; however the fact that we are able to describe the set algebraically even though we are unable to guess the geometric description, shows the power of the methods of analytic geometry.
Example 3: Describe the set of points the sum of whose distances from two perpendicular lines is 1.
Solution: Choose the perpendicular lines to be the coordinate axes. Let P(x,y) be any point with the required property.

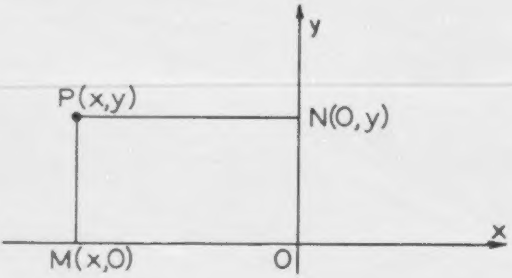


Fig. 2-6a

Then the distances of P from the perpendicular lines are

$$d(N,P) = \sqrt{(x - 0)^2 + (y - y)^2} = \sqrt{x^2} = |x|,$$

$$d(M,P) = \sqrt{(x - x)^2 + (y - 0)^2} = \sqrt{y^2} = |y|.$$

The geometric condition can now be written

$$d(N,P) + d(M,P) = 1$$

$$\sqrt{x^2} + \sqrt{y^2} = 1$$

or

$$|x| + |y| = 1.$$

If we want to sketch the graph of the set of points whose coordinates satisfy this equation, we might use the methods developed in Section 2-4.

The intercepts are (0,1), (0,-1), (1,0), (-1,0).

The tests for symmetry tell us that the graph is symmetric with respect to both axes and the origin. Hence if we plot the part of the graph in the first quadrant we can sketch the rest by symmetry. If $x \geq 0$, $y \geq 0$, then the equation can be written $x + y = 1$. The part in the first quadrant is shown in Figure 2-6b. The complete point set is shown in Figure 2-6c.

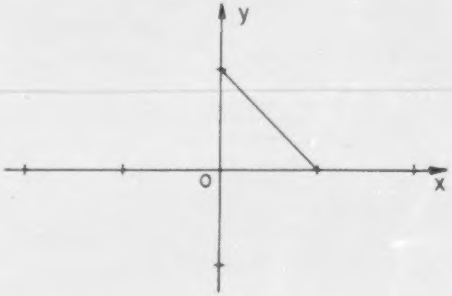


Fig. 2-6b

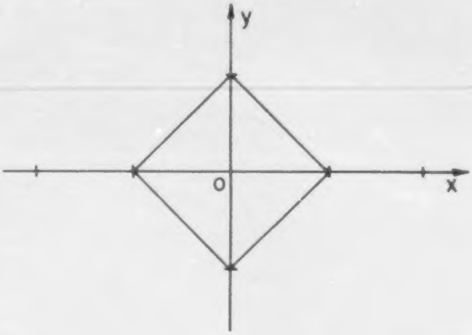


Fig. 2-6c

We shall use this algebraic technique for describing sets satisfying geometric conditions extensively in Chapter 6 when we make a systematic study of equations of the first and second degree in x and y.

Exercises 2-6

In each exercise the point set should be plotted.

1. Write the equation describing the set of points which are at a distance 2 from the origin.
2. Write the equation of the set of all points which are at a distance 1 from the point $C(1,0)$.
3. Write the equation of the set of all points which are at a distance 3 from the point $C(0,2)$.
4. Write the equation of the set of all points which are at a distance 5 from the point $C(2,3)$.
5. Write the equation of the set of all points which are k units from the point $C(-1,3)$.
6. Write the equation of the set of all points at a distance r from the point $C(h,k)$. Describe this set geometrically.
7. Write the equation of the set of all points which are equidistant from the points $A(3,0)$ and $B(5,0)$.
8. Write the equation of the set of all points which are equidistant from the points $A(-2,-5)$ and $P(3,2)$.

*9. Write the equation of the set of all points which are equidistant from the points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$. Describe this set geometrically.

10. Write the equation of the set of points each of which is twice as far from $A(-2,0)$ as it is from $B(1,0)$.
11. Write the equation of the set of points each of which is the vertex of a right triangle whose hypotenuse is the line segment joining $(-1,0)$ and $(1,0)$. Describe this set geometrically.

*12. Write the equation of the set of points each of which is the midpoint of a line segment of length 2 having its endpoints on two perpendicular lines.

13. Write the equation of the set of points each of which is the center of a circle which is tangent to the x-axis and which passes through the point $(0,1)$.

14. Write an equation whose only solution is $x = 0$, $y = 0$; that is, give an equation for the origin.
15. Write an equation for the semicircle of radius 2 with center at $(0,0)$ and lying to the left of the y-axis.
16. Write an equation of the set of all points (x,y) such that the area of the triangle with vertices (x,y), $(0,0)$ and $(3,0)$ is 2.

2-7. Supplementary Exercises for Chapter 2.

1. Discuss the symmetry and the intercepts of the graph of each equation.
   - (a) $x = 5y - 2$
   - (b) $3x^2 + 3y^2 = 12$
   - (c) $2x^2 - y = 3$
   - (d) $2x^2 + 3y^2 = 18$
   - (e) $\frac{x^2}{3} - \frac{y^2}{4} = 4$
   - (f) $y = -x^2 + 7x - 6$
   - (g) $y = \frac{2}{x - 1}$
   - (h) $x = y^2 + y$
   - (i) $y^2 - x^2 = 16$
   - (j) $y = -2x + 3$
   - (k) $x = |y|$
   - (l) $y = 2x^2 + 4$
   - (m) $16x^2 + 9y^2 = 144$
   - (n) $x^2 + 6x + y^2 = 7$
   - (o) $y = (x - 1)(x - 2)^2$
   - (p) $|x + y| = 0$
2. Sketch the graph of each of the above equations.
3. (a) Describe a line parallel to the y-axis in terms of coordinates.
   (b) Similarly, for the x-axis.
4. Sketch the graph of the following:
   (a) $\{(x,y) : x^2 + y^2 = 9\}$. [Read "x and y such that $x^2+y^2 = 9$".]
   (b) $\{(x,y) : x^2 + y^2 \geq 9\}$.

(c) $\{(x,y) : x^2 + y^2 < 9\}$.

(d) $\{(x,y) : |x| + |y| = 9\}$.

(e) $\{(x,y) : |x| + |y| > 9\}$.

(f) $\{(x,y) : (x \geq 0 \text{ and } y \geq 0) \text{ and } (|x| + |y| \leq 9)\} \cup$
$\{(x,y) : (x \leq 0 \text{ and } y \leq 0) \text{ and } (|x| + |y| \leq 9)\}$.

(g) $\{(x,y) : |x| + |y| < 9\} \cap \{(x,y) : x^2 + y^2 > 9\}$.

5. (a) Plot the points A(0,-3), B(-2,1) and C(6,5) and connect them with lines.
   (b) Show that the triangle formed is a right triangle.
   (c) Find the slope of the hypotenuse.
   (d) Find the area of the triangle.
6. Given the points A(6,2), B(8,-6) and C(10,0)
   (a) Find the distance between the midpoint of $\overline{AB}$ and $\overline{AC}$.
   (b) Find slope of line through the midpoint of $\overline{AB}$ and $\overline{BC}$.
7. Given the points A(2,4), B(4,-2) and C(-3,-1). What kind of triangle is ABC?
8. Given the points A(2,-3), B(-1,2) and C(a - 1, a - 3), D(2a,3a)
   (a) Find the value of a for which the line CA will be $\perp$ to the line CB.
   (b) Find the value of a for which the line CD will be parallel to the line AB.
9. Find the equation of the set of all points equidistant from A(0,0) and B(6,3).
10. Find the equation of the set of all points whose distance from point A(2,0) is 3 units and for which $y > 0$.
11. Prove the diagonals of a rectangle are equal in length.
12. Find the other end of a line segment if one end is (-4,8) and the midpoint is $(\frac{1}{2},1\frac{1}{2})$.
13. Plot the points A(-3,2), B(5,-2), C(10,10). Show that the line segment joining the midpoints of $\overline{AC}$ and $\overline{BC}$ is parallel AB and its length is equal to one-half the length of AB.

14. Determine y so that the point $P(1,y)$ lies on the perpendicular bisector of the line segment joining the points $A(3,2)$ and $B(7,6)$.
15. Write the equation of the set of all points
    (a) a distance 7 from the x-axis.
    (b) a distance 7 from the y-axis.
    (c) a distance 7 from the origin.
    (d) a distance 7 from the x-axis and a distance 7 from the y-axis.
16. Show that the point $C(6,3)$ is on the perpendicular bisector of the line segment whose endpoints are $A(3,2)$ and $B(7,6)$.
17. A circle with center at the origin passes through the point (a,b). Which of the following points is on the circle?
    (a) $(-a,-b)$
    (b) $(a,-b)$
    (c) $(-a,b)$.

    Explain.
18. Write a set description of the set of all points 3 units from the origin in which the set is restricted to
    (a) the first quadrant.
    (b) the second quadrant.
    (c) the second or third quadrants.
    (d) the first or third quadrants.
19. Write a set description of the set of points inside the triangle formed by the axes and the line $2x + y = 3$.
20. Write the equation of the set of points which is the vertex of an isosceles triangle whose base is the line between the points $A(-3,5)$ and $B(4,-1)$.
21. A line segment of variable length has its endpoints on the coordinate axes, forming with them a triangle whose area is constant. Write the equation of the set of midpoints of the segment.

22. Find the slope of the line which is tangent to the circle $x^2 + y^2 = 25$ at the point $P(-4,3)$.

23. Sketch the graph of the following set of points:

    (a) $\{(x,y) : y = x^2 - 4\}$

    (b) $\{(x,y) : y \leq 0 \text{ and } y = x^2 - 4\}$

    (c) $\{(x,y) : x \leq 0 \text{ and } y = x^2 - 4\}$

    (d) $\{(x,y) : (y \geq 0 \text{ and } x \geq 0) \text{ and } (y = x^2 - 4\}$

24. Plot the points $A(0,-3)$, $B(-2,1)$ and $C(6,5)$. Connect these points with lines forming the triangle ABC. Plot A', B', and C', their reflections in the x-axis. Connect these points with lines forming the triangle A'B'C'. Compare the areas and the perimeters of $\triangle ABC$ with those of $\triangle A'B'C'$.

Challenge Problems

1. Derive a formula which divides the line segment P_1P_2 in the ratio $r_1 : r_2$. Use this information to prove the medians of a triangle intersect in a point that is $\frac{2}{3}$ the distance from a given vertex to the midpoint of the opposite side.
2. Given the points $A(1,-2)$, $B(5,4)$ and $C(-3,4)$. Determine the coordinates of the centroid of $\triangle ABC$.
3. Suppose that a rectangular grid is constructed so that the units marked off on the x-axis are twice as long as those on the y-axis. Develop a suitable formula for the distance between any two points $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ in this coordinate system.
4. Design a new coordinate system such that the first of an ordered pair of numbers represents the slope of a line passing through the origin, and the second the length of the line. By convention + slope will mean a line rising to the right and - slope will mean a line rising to the left.

A + line length will mean above the horizontal and a - line length below the horizontal. Let the ordered pairs of numbers be represented by the letters s and d such that any point P can be represented as P(s,d).

Questions:

(a) Can you find the equation of a circle in this coordinate system? (Remember--a graph is a set of points each of which satisfies the equation.)

(b) What is the equation of a straight line passing through the origin?

(c) Draw the graph of $d = ks$, where k is a constant.

(d) Find the equation of a vertical line that does not pass through the origin. (Hint: Use the perpendicular distance p from the origin to the line and the Pythagorean Theorem.) Ans. $d = p\sqrt{s^2 + 1}$.

(e) See if you can find the equation of any line.

Chapter 3

THE FUNCTION CONCEPT AND THE LINEAR FUNCTION

3-1. Informal Background of the Function Concept.

The function concept is one of the most basic concepts of all mathematics and this whole chapter is devoted to the study of that important idea. We first try to form some idea of what the concept is about in an informal way.

We base our discussion of functions on sets. Mathematicians studied functions long before they talked about sets but they were led to formulate the function idea in terms of sets in order to make their study of this topic as clear as possible.

In order to have a function three things are required: a set called its domain; a set called its range; and a rule for pairing a member of the range with each member of the domain.

Example 3-1: Multiplying integers by 2 gives us an example of a function. The domain of this function is the set of all integers. The range of the function is the set of all even integers.

If you have already studied functions you have probably considered only those functions which pair numbers with numbers. For the functions we are studying now neither the domain nor the range has to be a set of numbers. The addition table for whole numbers defines a function whose domain is not a set of numbers. It assigns a whole number to each pair of whole numbers, namely their sum. The domain of this function is the set of all pairs of whole numbers. Its range is the set of all whole numbers. For instance the addition function assigns 19 to the pair (11,8) and 26 to the pair (12,14).

Exercises 3-1

Each of the following phrases suggests a function. Describe its domain, its range, and its rule.

1. Areas of triangles
2. The multiplication table for positive integers
3. Election returns
4. People's first names
5. People's ages
6. Population of cities
7. A dictionary
8. The relative nearness to the sun of the various planets
9. Batting averages
10. Absolute values

Give some examples of everyday circumstances which suggest functions.

3-2. Formal Definition of Function.

Definition 3-2a: Let A and B be sets and let there be given a rule which assigns exactly one member of B to each member of A . Then the rule, together with the set A is said to be a function and the set A is said to be its domain. The set of all members of B actually assigned to members of A by the rule is said to be the range of the function.

The word "rule" must be understood here to cover many different kinds of schemes for making assignments. Sometimes a rule is an algebraic expression, but sometimes it can be just a list of arbitrary pairings with no underlying pattern. Example 3-3a illustrates the first kind and Example 3-5b illustrates the second kind. Both the domain of a function and the range of a function are sets but they are subjected to quite different regulations. Every member of the domain of a function has exactly one member of the range assigned to it. However an individual member of the range of a function can be assigned to several different members of its domain.

The definition of function gives no instructions about which sets are to be used in the construction of functions. It gives no clues as to how to find the assignments that it mentions. We have to go beyond the definition to show how sets are selected and how rules are made to obtain useful and interesting functions.

Example 3-2a: The Constant Function. Let A be the set of all real numbers and let b be any real number. Then assigning b to each real number gives a function whose domain is the set A and whose range is $\{b\}$. Any such function whose range contains exactly one member is called a *constant function*.

Example 3-2b: The Identity Function. Let A be the set of all real numbers and assign each member of A to itself. These assignments constitute a function whose domain is A and whose range is A. Any such function which assigns each member of the domain to itself is called an *identity function*.

Example 3-2c: Multiplication Regarded as a Function. Let A be the set of all pairs (x,y) of real numbers and assign to each number (x,y) of A the product xy. These assignments constitute a function whose domain is A and whose range is the set of all real numbers.

Exercises 3-2

1. Each of the following defines a function. Describe its domain and range.
   - (a) Assign to each real number x the number $x + 2$.
   - (b) Assign to each real number x the number $5x$.
   - (c) Assign to each real number x the number $|x|$.
   - (d) Assign to each real number x the number x^2 .
   - (e) Assign to each real number x the number $(x + 15)^2$.
   - (f) Assign to each real number x the number 4 .
   - (g) Assign to each even integer the number 0 and to each odd integer the number 1 .
   - (h) Assign to each point in the plane the point 2 units to the right and 3 units down.
   - (i) Assign to each rectangle its area.
   - (j) Assign to each pair of distinct points in the plane the distance between them.
2. Let A be $\{1,2,3\}$ and let B be $\{4,5\}$.
   - (a) Define a function whose domain is A and whose range is B .
   - (b) Define a function whose domain is A, whose range is A, and which is not the identity function.
   - (c) Define a function whose domain is B and whose range is B .
   - (d) Show that there is no function whose domain is B and whose range is A .

3-3. Notation for a Function.

It is customary to denote functions by single letters such as f, g and h . If x is any member of the domain of function f, then $f(x)$ means the element assigned to x by the function f .

Note: The expression $f(x)$ is read "f of x" . Some people prefer to read it "f at x" .

Example 3-3a: Let the function f have for its domain the set of all real numbers, for its range the set of all non-negative real numbers, and for its rule the assignment to each real number of its square. Then $f(2) = 4$, $f(3) = 9$, $f(0) = 0$, $f(-3) = 9$, $f(x) = x^2$.

Sometimes this notation is used to cover more complicated situations. By $f(g(x))$ we mean the expression obtained by substituting $g(x)$ for x in $f(x)$.

Example 3-3b: If $f(x) = 3x$ and $g(x) = 2x$ then

$$f(g(x)) = 3(2x) = 6x$$

Example 3-3c: If $f(x) = 3x^2 + 2$ and $g(x) = 4x - 1$ then

$$f(g(x)) = 3(4x - 1)^2 + 2$$

The idea here is that if it makes sense to substitute an expression E for x in $f(x)$ then the symbol $f(E)$ is used to describe the result of performing this substitution.

Example 3-3d: If $f(x) = x^2 - 3x + 2$ then

$$f(2h + 4) = (2h + 4)^2 - 3(2h + 4) + 2$$

Exercises 3-3

1. Given that f is the function whose domain is the set of all positive integers $\{1,2,3, \ldots\}$ and which pairs with each integer x the integer $3x$. (a) What is the range of f? (b) $f(4) = ?$, (c) $f(6) = ?$, (d) $f(a) = ?$, (e) $f(3a) = ?$, (f) $f(2 + x) = ?$, (g) Does $f(3x) = 3f(x)$? (h) Does $f(3x + 4) = 3f(x) + 4$?

2. Given that f is the function whose domain is the set of all positive integers, $\{1,2,3, \ldots\}$ which assigns 0 to the even integers and 1 to the odd integers. (a) What is the range of f ? (b) $f(2) = ?$ (c) $f(3) = ?$ (d) $f(10^4) = ?$
   (e) Does $f(3) + f(5) = f(3 + 5)$?
   (f) Does $f(3) + f(4) = f(3 + 4)$?
   (g) Does $f(2) + f(4) = f(2 + 4)$?
   (h) Does $f(3) \cdot f(4) = f(3 \cdot 4)$?
   (i) Does $f(2) \cdot f(4) = f(2 \cdot 4)$?
   (j) Does $f(x + 2) = f(x)$?
   (k) Does $f(x + 1) = f(x)$?
   (l) Does $f(x \cdot 2) = f(x)$?
3. Let f be a function whose domain is $\{x : -1 < x < 2\}$. If $f(x) = |x|$, what is the range of f ?
4. Let f be a function whose domain is the set of all real numbers. If $f(x) = |x| - x$, what is the range of f ?

3-4. Functions Defined by Equations.

Many of the functions we shall meet have sets of real numbers for their domain and range, and have rules which are expressed by algebraic equations. For instance the function defined in Example 3-3a is such a function. These special functions are often defined only by giving the rule, with no mention of the domain and range. This causes no confusion if the student knows how to supply the proper domain and range himself. Usually in what follows, if a function is discussed whose domain is not given explicitly, its domain is understood to be the set of all real numbers. For instance, the equation $y = 3x^2 + 1$ can be used to define a function whose domain is the set of all real numbers, whose range is $\{y : y \geq 1\}$ and whose rule is to pair with each real number x the number $3x^2 + 1$. It is customary to express all this information in more compact form by referring to "the function defined by the equation $y = 3x^2 + 1$" .

If we apply this agreement to the equation $y = \frac{3}{x - 4}$ we see that this equation does not define a function whose domain is the set of all real numbers. The right member of this equation is meaningless for $x = 4$. Nevertheless, the equation $y = \frac{3}{x - 4}$ can be used to define a function provided a set of real numbers which does not contain the number 4 is specified as its domain. We therefore modify our agreement. When we encounter an equation of the form $y = f(x)$ we assume that the function it defines has for domain the set of all real numbers which can meaningfully be substituted for x in $f(x)$, unless some other domain is given explicitly. For instance "the function defined by the equation $y = \frac{3}{x - 4}$" means the function whose domain is the set $\{x : x < 4 \text{ or } x > 4\}$, whose range is $\{y : y < 0 \text{ or } y > 0\}$ and whose rule is to pair $\frac{3}{x - 4}$ with each number x in its domain.

Exercises 3-4

1. Let f be the function defined by the equation $y = 2x + 6$.
   (a) What is the domain of f ? (b) What is the range of f ?
   (c) $f(2) = ?$ (d) For what value of x does $f(x) = 100$?
   (e) For what value of x does $f(x) = 0$?
2. What is the domain and range of the function defined by each of the following equations
   (a) $y = 3x$
   (b) $y = \frac{3}{x}$
   (c) $y = \sqrt{x}$
   (d) $y = x^3$
   (e) $y = \sqrt[3]{x}$
3. Let f be the function defined by the equation $y = x^2$.
   (a) What is the domain of f ? (b) What is the range of f ?
   (c) Is there a number x such that $f(x) = 6$?
   (d) Is there a number x such that $f(x) = -6$?

4. Let f be the function defined by the equation $y = x^3$.
   (a) What is the domain of f ?
   (b) What is the range of f ?
   (c) Is there a number x such that $f(x) = 6$?
   (d) Is there a number x such that $f(x) = -6$?

5. Let n be a positive integer and let f be the function defined by the equation $y = x^n$.
   (a) What is the domain of f ?
   ~~(b) What is the range of f ?~~
   (c) Is there a number x such that $f(x) = 6$?
   (d) Is there a number x such that $f(x) = -6$?

6. Let f be the function defined by the equation $y = \frac{1}{x}$.
   (a) What is the range of f ?
   (b) What is the domain of f ?
   (c) Is there a number x such that $f(x) = 6$?
   (d) Is there a number x such that $f(x) = -6$?

7. Let f be the function defined by $y = \frac{1}{x^2}$.
   (a) What is the range of f ?
   (b) What is the domain of f ?
   (c) Is there a number x such that $f(x) = 6$?
   (d) Is there a number x such that $f(x) = -6$?

8. Let f be the function defined by $y = \frac{1}{x^n}$ where n is a positive integer.
   (a) What is the range of f ?
   (b) What is the domain of f ?
   (c) Is there a number x such that $f(x) = 6$?
   (d) Is there a number x such that $f(x) = -6$?

3-5. The Graph of a Function.

Two sets are needed to define a function, one to be the domain and one to be the range. After the function is defined a new set is created, namely the set of all those pairs produced by the rule of the function. This set is sometimes called the graph of the function. Indeed many mathematicians claim that this set is the function itself.

Example 3-5a: Let f be the function defined by the equation $y = 4x - 7$. Then its graph consists of all the ordered pairs of the form $(x, 4x - 7)$. For instance $(0,-7)$, $(1,-3)$, (2,

are some of the pairs of this graph.

Example 3-5b: Let f be the function whose domain is $\{1,2,3\}$, whose range is $\{6,5\}$, and whose rule assigns 6 to 1, 5 to 2, 6 to 3. Then the graph of f is the set $\{(1,6), (2,5), (3,6)\}$. This is a function whose rule has no pattern. It was constructed by making arbitrary pairings.

If a function happens to have a domain and range consisting of real numbers then the pairs of its graph can be plotted as points. The resulting geometric figure is also called the "graph of the function". This implies that the expression "the graph of a function" can mean two different things. The more usual meaning is the geometric figure. The graph of a function defined by an equation is generally considered to be the same as the graph of that equation, as defined in Chapter 2.

Most of the functions defined by algebraic equations have smooth curves as their graphs. The student almost always has to rely on this fact in order to draw the graph of a function.

Example 3-5c: Try to draw the graph of the function defined by $y = 3x^2 + 1$.

Solution: Choose several values of x and compute the numbers assigned to these values by the function.

x	$3x^2 + 1$
-3	28
-2	13
-1	4
0	1
1	4
2	13
3	28

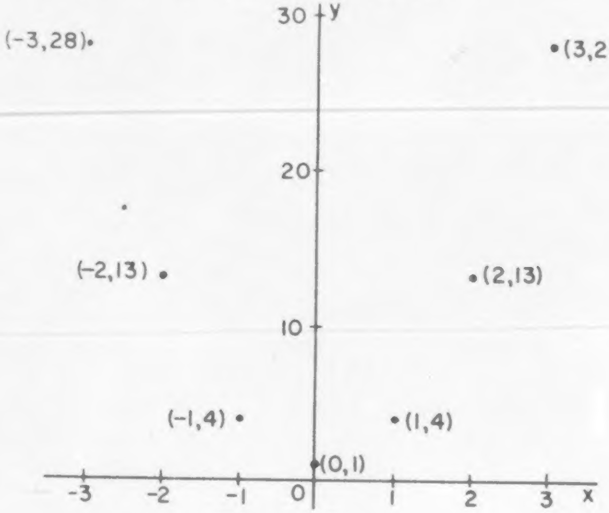


Fig. 3-5a

We find that 28 is assigned to -3, 13 to -2, 4 to -1, 1 to 0, 4 to 1, 13 to 2, 28 to 3 . Thus we know that part of the graph looks like Figure 3-5a. We can fill in the rest of the graph as in Figure 3-5b if we believe that the graph is a smooth curve.

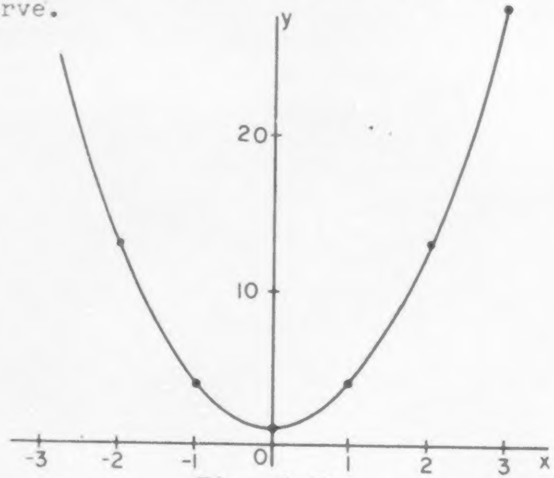


Fig. 3-5b

[sec. 3-5]

We do not now have a logical reason for excluding the curve in Figure 3-5c as the graph.

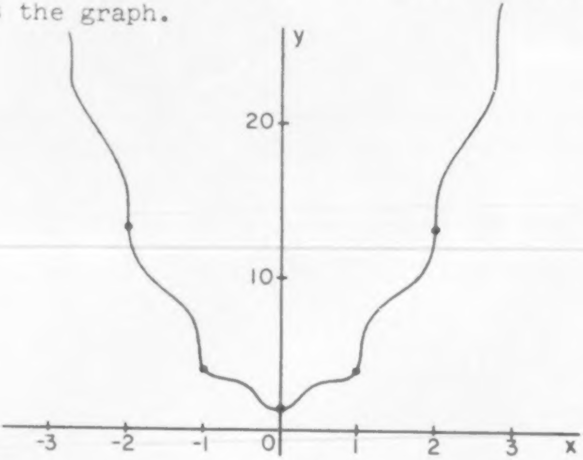


Fig. 3-5c

Later we shall prove that it is Figure 3-5b which is the correct one rather than Figure 3-5c.

Exercises 3-5

1. Can the pairs $(1,2)$ and $(1,3)$ occur in the graph of the same function? Justify your answer.
2. Can the pairs $(2,1)$ and $(3,1)$ occur in the graph of the same function? Justify your answer.
3. Plot the graph of the functions defined by each of the following equations:
   (a) $y = -2x + 1$
   (b) $y = -3x - 2$
   (c) $y = x - 2$
   (d) $y = +2x + 3$
4. Plot the graph of the functions defined by each of the following equations:
   (a) $y = -x^2 + 6$
   (b) $y = 2x^2 - 1$
   (c) $y = -x^2 - 3$
   (d) $y = 3x^2 + 2$
   (e) $y = x^3$

3-6. Functions Defined Geometrically.

We are going to use some of the facts of Coordinate Geometry to introduce another way of defining functions. Every point of the plane pairs two numbers, its x-coordinate and its y-coordinate. For some sets of points these pairings are the pairings of a function. For instance if no two points in a set have the same x-coordinate then assigning the y-coordinate of each point of this set to its x-coordinate defines a function. The domain of the function is the set of all x-coordinates of points of the set. The range of the function is the set of all y-coordinates of the set. It is also the case that if no two members of the set have the same y-coordinate, then assigning the x-coordinate to the y-coordinate of each point of the set defines a function. However we follow the generally accepted practice of using only the first scheme for defining functions. Thus we shall consider that a set of points defines a function if and only if no two points of the set have the same x-coordinate. The geometrical way of stating this condition is that a set of points defines such a function if and only if no vertical line contains more than one point of the set.

Example 3-6a: In Figure 3-6a the figure consisting of the three points $(-1,3)$, $(1,3)$ and $(2,3)$ defines a function f

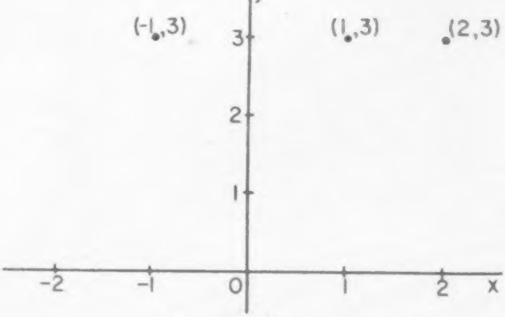


Fig. 3-6a

whose domain is $\{-1,1,2\}$, whose range is $\{3\}$, and whose rule makes the assignments $f(-1) = 3$, $f(1) = 3$, $f(2) = 3$.

Example 3-6b: The graph of the equation $y = 3x^2 + 1$ defines a function whose domain is the set of all real numbers

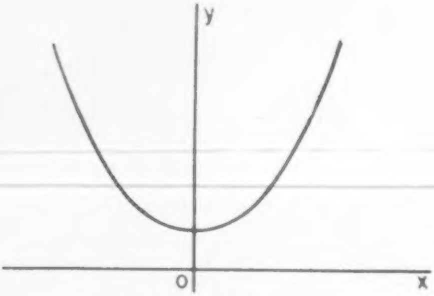


Fig. 3-6b

and whose range is $\{y : y \geq 1\}$.

Example 3-6c: The curve sketched in Figure 3-6c defines a

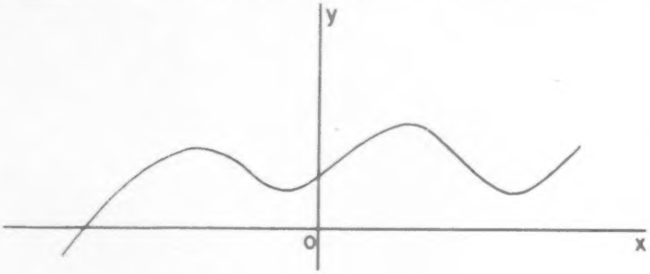


Fig. 3-bc

function. The curve has no simple equation.

Example 3-6d: The part of the graph of the equation

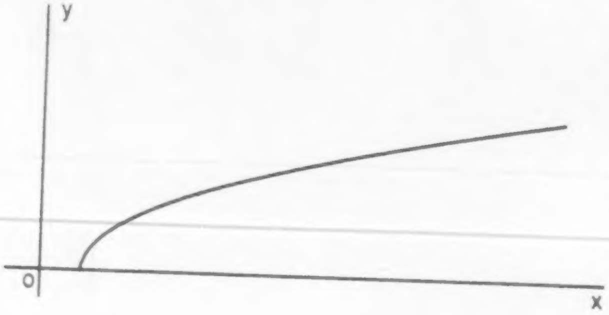


Fig. 3-6d

$x = 3y^2 + 1$ sketched in Figure 3-6d defines a function.

In each of the following examples notice that there is a vertical line that intersects the graph in more than one point.

Example 3-6e: The graph of the equation $x = 3y^2 + 1$

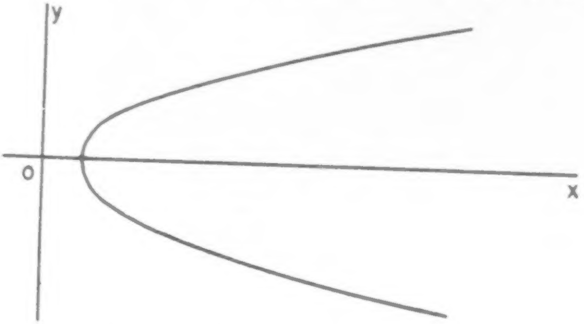


Fig. 3-6e

does not define a function.

Example 3-6f: The circle whose center is (0,0) and

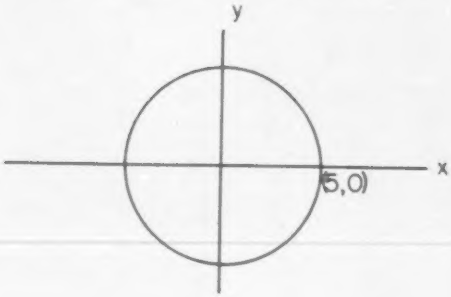


Fig. 3-6f

whose radius is 5 does not define a function.

Example 3-6g: The Figure 3-6g consisting of the points

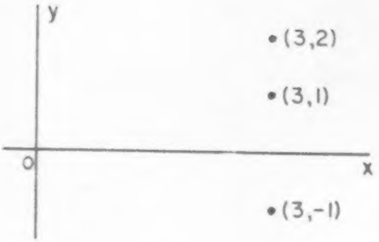


Fig. 3-6g

(3,-1), (3,1), (3,2) does not define a function.

Example 3-6h: The set of points in Figure 3-6h does not

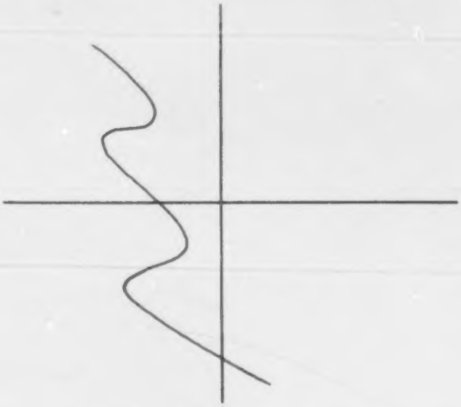


Fig. 3-6h

define a function.

Exercises 3-6

1. Can a circle be the graph of a function?
2. Can a semi-circle be the graph of a function?
3. Are there semi-circles which are not the graphs of functions?
4. Can a triangle be the graph of a function?
5. Can a line be the graph of a function?
6. Are there lines which are not graphs of functions?
7. Which of the following are graphs of functions? Justify your answer.

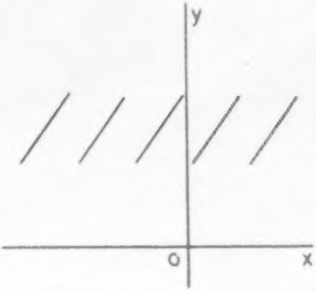


Fig. 3-6(7a)

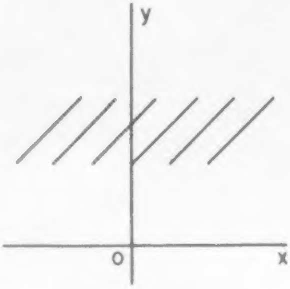


Fig. 3-6(7b)

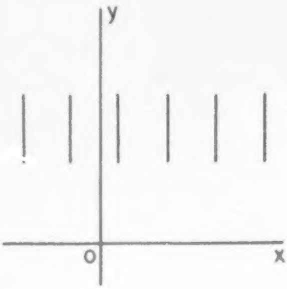


Fig. 3-6(7c)

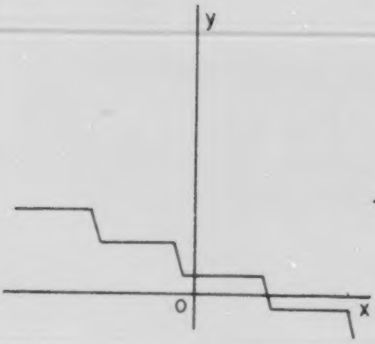


Fig. 3-6(7d)

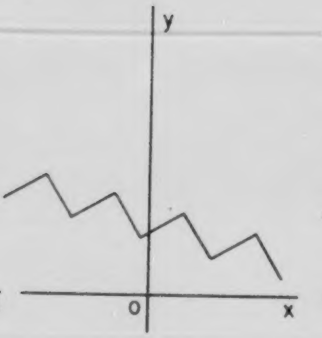


Fig. 3-6(7e)

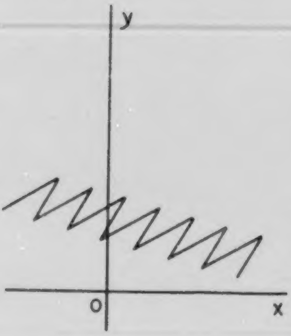


Fig. 3-6(7f)

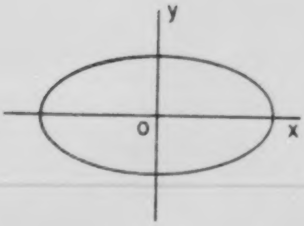


Fig. 3-6(7g)

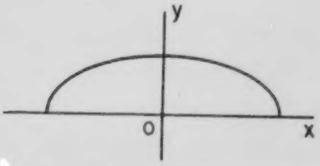


Fig. 3-6(7h)

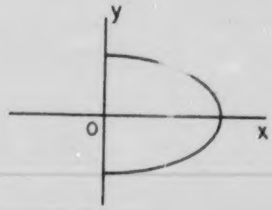


Fig. 3-6(7i)

8. Prove that if n is an odd integer then the graph of $y^n = x$ defines a function and that if n is an even integer then the graph of $y^n = x$ does not define a function.

3-7. Functions Defined by Physical Processes.

Someone who understands the function concept can find examples of functions in every aspect of his daily life. While this does not always help people to understand what is going on around them, the discovery and study of such functions is an important part of any scientific analysis of our world.

Example 3-7a: A falling body defines many functions. For example, at each instant, a falling body has a speed, and pairing speed with time produces a function. Physicists have discovered that for a body falling from rest in a vacuum, the equation $y = 32t$ defines this function, where t is the number of seconds after the body began to fall and y is its speed in feet per second. Another function defined by the falling body is the one which pairs the distance it falls with the elapsed time. Physicists have discovered that the equation which defines this function is $y = 16t^2$, where t is the number of seconds after the body begins to fall and y is the number of feet the body falls in t seconds.

Example 3-7b: The mass of a radioactive body decreases with time. Such a body defines a function; assign to each instant of time the mass of the body at that instant. When we study the exponential function, in Chapter 9, we shall see an equation that defines this function.

Exercises 3-7

Below are some descriptions of physical situations which define functions. Try to find the domain and range for each. Express the rule in algebraic form if you know it; try to make a reasonable guess if you don't.

1. If a gas is kept at constant temperature its volume and its pressure are dependent on each other.
2. The time it takes a pendulum to complete a swing depends on the length of the pendulum.
3. The gravitational attraction of the earth on a body depends on the body's distance from the earth.
4. If the ends of a beam are clamped and if an object is hung on it the distance the beam is displaced depends on the weight of the object.
5. The apparent brightness of a light source to an observer depends on the distance of the observer from the source.
6. The force exerted by a lever depends on the distance of its end from the fulcrum.
7. If water is flowing at a uniform rate through a pipe into a tank, the amount of water in the tank depends on the time of flow.
8. The temperature of a cup of coffee depends on the time it has been cooling.
9. The temperature at which water boils depends on altitude.
10. The time it takes an automobile to come to a halt depends on its speed.

3-8. Functions Defined by Composition; Inverses.

Functions can sometimes be defined in terms of other functions. This is so, for instance, if f and g are functions for which the range of f is the domain of g.

Example 3-8a: Let f be defined by $y = 3x - 5$ and g be defined by $y = 3x^2 + 1$. Then the equation $y = 3(3x - 5)^2 + 1$ defines a new function.

The following definition is a formal statement of this procedure.

Definition 3-8a: Let A, B and C be sets, let f be a function whose domain is A and whose range is B, let g be a function whose domain is B and whose range is C. Then by the composition of g with f we mean the function whose domain is A, whose range is C and which assigns to each member x of A, the member $g(f(x))$ of C.

Example 3-8b: Let A be the set $\{4,5,6\}$, let B be the set $\{7,8\}$ and let C be the set $\{9,10\}$, let $f(4) = 8$, $f(5) = 7$, $f(6) = 8$ and let $g(7) = 10$, $g(8) = 9$. Then the graph of the composition of g with f is $\{(4,9), (5,10), (6,9)\}$.

It is sometimes helpful to imagine that the rule of a function describes an action which does something to each member of the domain to produce the corresponding member of the range. From this point of view it is possible also to imagine a process which undoes what the original function does. The function defined by equation $y = x + 6$ has the effect of adding 6 to each number. The function defined by the equation $y = x - 6$ has the effect of subtracting 6 from each number. Thus each of these functions "undoes" what the other does. The definition of inverse function which follows expresses these ideas formally.

Definition 3-8b: Let A and B be sets, let f be a function whose domain is A and whose range is B and let g be a function whose domain is B and whose range is A. Then we say that f and g are inverse functions if, for each x of A, $f(g(x)) = g(f(x)) = x$. We also say that f is the inverse of g and g is the inverse of f if f and g are inverse functions.

Example 3-8c: Let A be the set of all integers, let f be the function which assigns to each integer x the integer $x + 1$ and let g be the function which assigns to each integer x the integer $x - 1$. Then $f(x) = x + 1$, $g(x) = x - 1$, $f(g(x)) = (x - 1) + 1 = x$ and $g(f(x)) = (x + 1) - 1 = x$. Therefore f and g are inverse functions.

Example 3-8d: Let A be the set of positive real numbers, let f have domain A and be defined by $y = x^2$, let g have domain A and be defined by $y = \sqrt{x}$. Then $g(f(x)) = \sqrt{x^2} = x$ and $f(g(x)) = (\sqrt{x})^2 = x$ which identifies f and g as inverses.

Some functions have no inverse. Consider, for example, the function f whose domain is the set of all real numbers defined by $y = x^2$. If this function had an inverse g, then since $f(2) = 4$, $f(-2) = 4$ we would have to have $g(4) = 2$ and $g(4) = -2$. But this is impossible, since a function must assign only one member of its range to a member of its domain. Notice that the single equation $y = x^2$ was used to define a function with an inverse and a function with no inverse.

Theorem 3-8a: Let f be a function whose domain is the set A and whose range is the set B. Then f has an inverse if and only if for each member b of B there is exactly one a of A for which $f(a) = b$.

Proof: Because B is the range of f, for each b of B there is at least one a of A such that $f(a) = b$. Assign to each b all such members a. These assignments define a function g, whose domain is B, if and only if, this rule pairs only one a with each b. If there is such a g, then

$g(f(a)) = g(b) = a$ for each a of A, so $g(f(x)) = x$. Also for each b of B, $f(g(b)) = f(a) = b$, so $f(g(x)) = x$. This identifies g as the inverse of f .

If f and g are inverses, their graphs are closely interrelated.

Theorem 3-8b: Let f and g be inverses. Then a pair (p,q) is in the graph of f if and only if (q,p) is in the graph of g .

Proof: If the pair (p,q) is in the graph of f, then $q = f(p)$. If g is the inverse of f then $g(q) = p$. Thus (q,p) is in the graph of g . Similarly if (q,p) is in the graph of g , then $p = g(q)$. If f is the inverse of g, then $f(p) = q$. Thus (p,q) is in the graph of f .

By plotting the graph of a function, it is possible to see whether the function has an inverse or not and also, if the function does have an inverse, to see what the graph of the inverse actually is. If no horizontal line has more than one point on the graph of a function, then the function has an inverse.

Example 3-8e: Figures 3-8a, 3-8b, 3-8c and 3-8d are graphs of functions. Figures 3-8a and 3-8b correspond to functions with

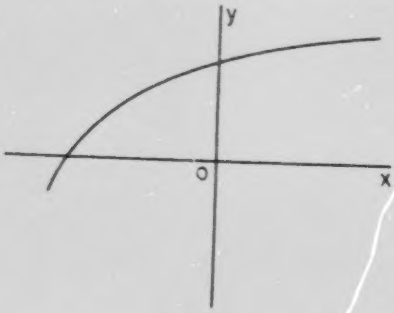


Fig. 3-8a

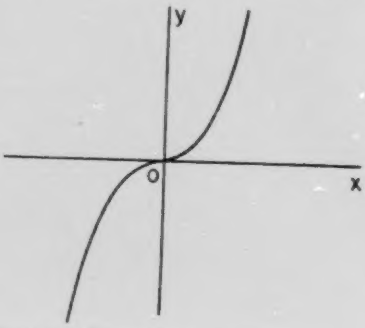


Fig. 3-8b

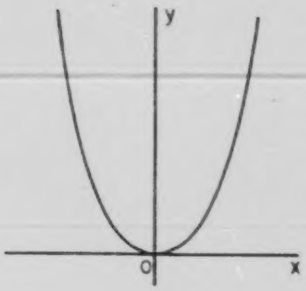


Fig. 3-8c

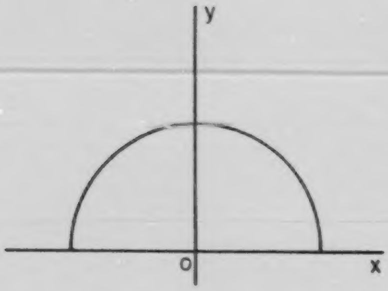


Fig. 3-8d

inverses. Figures 3-8c and 3-8d correspond to functions without inverses. Notice that every horizontal line intersects the graphs of Figures 3-8a and 3-8b in a single point. Notice that some horizontal line intersects the graphs of Figures 3-8c and 3-8d in 2 points.

Again Figures 3-8e and 3-8f show the graphs of the inverses of the function associated with Figures 3-8a and 3-8b. Notice

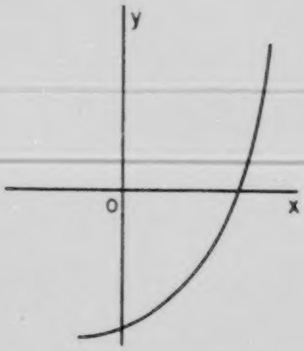


Fig. 3-8e

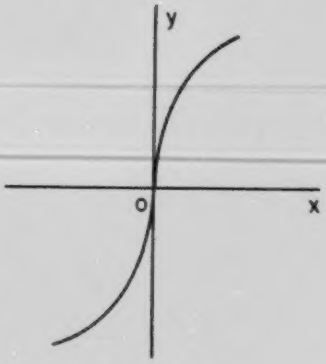


Fig. 3-8f

that Figures 3-8e and 3-8f are obtained from Figures 3-8a and 3-8b by interchanging the x and y coordinates. This illustrates Theorem 3-8b.

Exercises 3-8

1. Let $f(x) = x^2$ and $g(x) = x^3$.
   (a) What is $f(g(x))$?
   (b) What is $g(f(x))$?
   (c) Does $f(g(x) = g(f(x))$?
2. Let $f(x) = x^2$ and $g(x) = x^3 + 1$.
   (a) What is $f(g(x))$?
   (b) What is $g(f(x))$?
   (c) Does $f(g(x) = g(f(x))$?
3. The function f is defined by $y = 2x + 3$. Show that its inverse is the function defined by $y = \frac{1}{2}x - \frac{3}{2}$.
4. The function f is defined by $y = 4x + 5$. Show that the function defined by $y = \frac{1}{4x + 5}$ is not the inverse of f .
5. Which of the functions defined in Exercises 3-6, Problem 7 has an inverse?
6. (a) Show that the points (a,b) and (b,a) are symetrically situated with respect to the line $y = x$. Show how to use this fact to find the graph of a function from the graph of its inverse.
   (b) Sketch the graph of the inverse of the function whose graph is shown in Figure 3-8(6a) - Figure 3-8(6d).

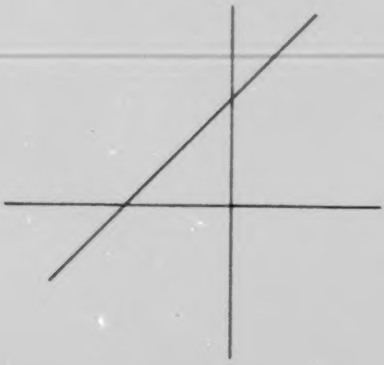


Fig. 3-8(6a)

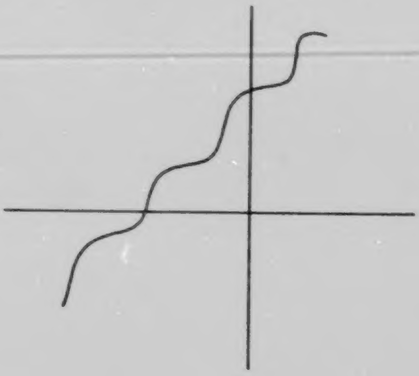


Fig. 3-8(6b)

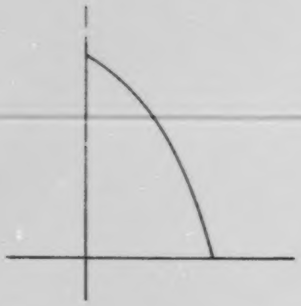


Fig. 3-8(6c)

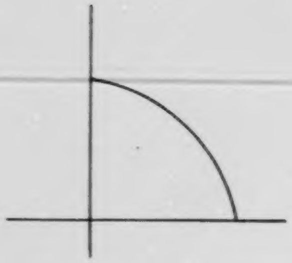


Fig. 3-8(6d)

3-9. The Linear Function.

Definition 3-9a: A function is a linear function if and only if it is defined by an equation $y = ax + b$, where a is a non-zero real number and where b is any real number.

Example 3-9a: Each of the following equations defines a linear function:

$y = 3x + 4$	$(a = 3,\ b = 4)$
$y = -5x + 6$	$(a = -5,\ b = 6)$
$y = x - 11$	$(a = 1,\ b = -11)$
$y = 2x$	$(a = 2,\ b = 0)$
$y = x$	$(a = 1,\ b = 0)$

Theorem 3-9a: Every linear function sets up a one-to-one correspondence between the set of all real numbers and the set of all real numbers.

Proof: Let the linear function f be defined by the equation $y = ax + b$. We are to show that

(1) if r is any real number then f assigns some real number $f(r)$ to r.

(2) if s is any real number then there is some real number t such that $s = f(t)$.

The first part is easy to prove; the number $ar + b$ is assigned by f to r.

To prove the second part we solve $ax + b = s$ for x, obtaining $x = \frac{s - b}{a}$. Then $f\left(\frac{s - b}{a}\right) = s$ because

$$f\left(\frac{s - b}{a}\right) = a\left(\frac{s - b}{a}\right) + b .$$

Therefore $\frac{s - b}{a}$ is a number t such that $f(t) = s$.

Corollary: Every linear function has an inverse.

It is proved in Chapter 6 that the graph of an equation $y = ax + b$ is a straight line. We can check this statement now with an example.

Example 3-9b: Plot the graph of $y = 3x + 2$.

Solution: We first construct a short table of values. We plot these points and obtain the part of the graph shown in Fig. 3-9a.

x	-2	-1	0	1	2	
y	-4	-1	2	5	8	

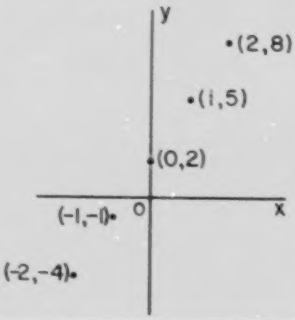


Fig. 3-9a

It certainly looks as though these points are collinear and it is not hard to believe that the line they determine is the graph of the function.

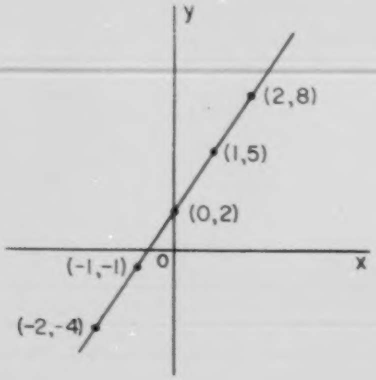


Fig. 3-9b

The graph of this function is the line shown in Figure 3-9b.

Theorem 3-9b: If f is the linear function defined by $y = ax + b$, its inverse g is the linear function defined by $y = \frac{1}{a}x - \frac{b}{a}$.

Proof: $f(g(x)) = a(\frac{1}{a}x - \frac{b}{a}) + b = x$ and

$$g(f(x)) = \frac{1}{a}(ax + b) - \frac{b}{a} = x \ .$$

Example 3-9c: Let f be defined by $y = 3x + 4$, then

$$g = \frac{1}{3}x - \frac{4}{3} \quad \text{and}$$

$$f(g(x)) = 3(\frac{1}{3}x - \frac{4}{3}) + 4 = x$$

$$\text{and} \quad g(f(x)) = \frac{1}{3}(3x + 4) - \frac{4}{3} = x \ .$$

Theorem 3-9b provides a formula for finding inverses of linear functions. It is probably easier not to use this formula to find the inverse of any given linear function, but rather to proceed as follows: If the function is defined by the equation $y = ax + b$

(1) solve the equation for x in terms of y ,

(2) interchange x and y in the answer.

Example 3-9c: (reworked in the recommended way)

$$\begin{aligned} y &= 3x + 4 \\ y - 4 &= 3x \\ x &= \frac{1}{3}y - \frac{4}{3} \end{aligned}$$

and interchanging x and y yields the equation

$$y = \frac{1}{3}x - \frac{4}{3} .$$

Linear functions pair real numbers with real numbers. The following two theorems show how the pairings made by linear functions are different from the pairings made by other types of functions. Theorem 3-9c states that linear functions have a certain property and Theorem 3-9d states that linear functions are the only functions which have this property.

Theorem 3-9c: Let the linear function f be defined by $y = ax + b$ $(a \neq 0)$ and let p and q be any distinct real numbers. Then

$$\frac{f(p) - f(q)}{p - q} = a$$

Proof:

$$\begin{aligned} \frac{f(p) - f(q)}{p - q} &= \frac{(ap + b) - (aq + b)}{p - q} \\ &= \frac{ap + b - aq - b}{p - q} \\ &= \frac{a(p - q)}{p - q} \\ &= a \end{aligned}$$

Example 3-9d: If f is defined by $y = 3x - 4$ and $p = 1961$, $q = 30$, then

$$f(p) = 3(1961) - 4 = 5879$$

$$f(q) = 3(30) - 4 = 86$$

$$f(p) - f(q) = 5793$$

$$p - q = 1931$$

$$\frac{f(p) - f(q)}{p - q} = 3 .$$

This theorem has a geometric interpretation. The points $(p,f(p))$ and $(q,f(q))$ are on the graph of the linear function f. According to Formula 2-3a of Chapter 2, the expression

$$\frac{f(p) - f(q)}{p - q}$$

is the slope of the line. Theorem 3-9c therefore has two consequences. One is that the graph of $y = ax + b$ has slope $\underline{a}$. The other is that this slope can be computed from the coordinates of any pair of distinct points on the line.

Theorem 3-9d: (Converse of Theorem 3-9c) Let t be any real number except zero, and let f be a function whose domain and range are the set of all real numbers. If for each pair of distinct real numbers p and q

$$\frac{f(p) - f(q)}{p - q} = t,$$

then f is a linear function.

Proof: Let q_0 be any real number. Then for every x

$$\frac{f(x) - f(q_0)}{x - q_0} = t$$

$$f(x) - f(q_0) = t(x - q_0)$$

$$f(x) = tx - tq_0 + f(q_0) .$$

Therefore $f(x)$ has a representation $ax + b$, where a is the given number t and b is the number $-tq_o + f(q_o)$.

Example 3-9e: Suppose $t = 6$ and $f(3) = 5$.

Then
$$\frac{f(x) - f(3)}{x - 3} = 6$$
$$\frac{f(x) - 5}{x - 3} = 6$$
$$f(x) - 5 = 6x - 18$$
$$f(x) = 6x - 13$$

Exercises 3-9

1. Which of the following equations defines a linear function?
   (a) $y = 7x + 2$
   (b) $y = 7x - 2$
   (c) $y = 7x$
   (d) $y = 2$
   (e) $y = x \cdot 6 + 4$
   (f) $y = \frac{x}{6} + 4$
   (g) $y = x^6 + 4$
   (h) $y = 0$
2. Let f be a linear function defined by $y = 5x + 6$.
   (a) $f(0) = ?$
   (b) $f(\frac{1}{2}) = ?$
   (c) $f(11) = ?$
   (d) For what values of x does $f(x) = 0$?
   (e) For what values of x does $f(x) = \frac{1}{2}$?
   (f) For what values of x does $f(x) = 11$?
3. Plot the graphs of all of the following equations on the same set of axes.
   (a) $y = 2x + 3$
   (b) $y = 2x - 3$
   (c) $y = 2x + 1$
   (d) $y = 2x$
   (e) $y = 2x - 5$

4. Plot the graphs of all of the following equations using a single set of axes.

   (a) $y = -4x + 2$
   (b) $y = x + 2$
   (c) $y = 3x + 2$
   (d) $y = -x + 2$
   (e) $y = -2x + 2$.

5. Plot the graphs of all of the following equations using a single set of axes.

   (a) $y = 5x + 6$
   (b) $y = -5x + 6$
   (c) $y = 5x - 6$
   (d) $y = -5x - 6$.

6. Each of the following equations defines a linear function. Find its inverse.

   (a) $y = 2x - 1$
   (b) $y = 3x + 5$
   (c) $y = -2x + 6$
   (d) $y = -x - 4$
   (e) $y = 6x + 7$.

7. For each of the functions of Problem 6 plot its graph and the graph of its inverse using a single set of axes for each pair.

8. The function f is defined by $y = 2x - 7$.

   (a) Find its inverse g.
   (b) $f(6) = ?$
   (c) $g(f(6)) = ?$
   (d) $g(6) = ?$
   (e) $f(g(6)) = ?$

9. The function f is defined by $y = -3x - 4$. Predict without computation the value of

$$\frac{f(1000) - f(100)}{1000 - 100}.$$

   Check your prediction by computation.

10. Plot the graph of $y = -3x - 4$. Pick two points on the graph, measure their coordinates, and use these values to compute the slope of the line.

11. Let f be defined by $y = x^2$. Show by direct computation that $\frac{f(3) - f(5)}{3 - 5} \neq \frac{f(4) - f(6)}{4 - 6}$.

12. (a) If f is defined by $y = x^2$, for how many values of x does $\frac{f(x) - f(7)}{x - 7} = \frac{f(9) - f(7)}{9 - 7}$?

(b) If f is a linear function for how many values of x does $\frac{f(x) - f(7)}{x - 7} = \frac{f(9) - f(7)}{9 - 7}$?

3-10. Linear Functions Having Prescribed Values.

Theorem 3-10a: Let x_1 and x_2 be any distinct real numbers and let y_1 and y_2 be any distinct real numbers. Then there is one and only one linear function f such that $y_1 = f(x_1)$ and $y_2 = f(x_2)$.

Proof: We seek real numbers a and b such that

$$ax_1 + b = y_1$$
$$ax_2 + b = y_2 .$$

To solve these equations subtract the second equation from the first, obtaining

$$a(x_1 - x_2) = y_1 - y_2 .$$

It follows that

$$a = \frac{y_1 - y_2}{x_1 - x_2} .$$

Substitute this expression for a in the first equation obtaining

$$y_1 = \frac{y_1 - y_2}{x_1 - x_2} x_1 + b$$

from which

$$b = y_1 - \frac{y_1 - y_2}{x_1 - x_2} x_1$$

or

$$b = \frac{x_1 y_2 - y_1 x_2}{x_1 - x_2} .$$

Then if there is a function which makes the given pairings it is defined by

$$y = \frac{y_1 - y_2}{x_1 - x_2}x + \frac{x_1y_2 - y_1x_2}{x_1 - x_2} .$$

It should be checked by direct substitution that this function actually makes the given pairings.

Example 3-10a: Determine the linear function for which $f(3) = 4$, $f(5) = -1$.

Solution: Determine a and b so that

$$4 = 3a + b$$
$$-1 = 5a + b .$$

Subtract to obtain $-2a = 5$, therefore $a = -\frac{5}{2}$. Substitute this result in the first equation to obtain

$$4 = 3(-\frac{5}{2}) + b, \text{ therefore}$$
$$b = \frac{23}{2} .$$

The required function is defined by $y = -\frac{5}{2}x + \frac{23}{2}$.

Check: $-\frac{5}{2} \cdot 3 + \frac{23}{2} = -\frac{15}{2} + \frac{23}{2} = \frac{8}{2} = 4$

$$-\frac{5}{2} \cdot 5 + \frac{23}{2} = -\frac{25}{2} + \frac{23}{2} = -\frac{2}{2} = -1 .$$

This theorem is closely related to the geometric fact that two points determine a line. The points in question are the points (x_1,y_1) and (x_2,y_2) . Theorem 3-10a says that if these points are not on a vertical line (that is $x_1 \neq x_2$) and not on a horizontal line (that is $y_1 \neq y_2$) then the line they are on is the graph of a linear function. Some students will feel that leaving out vertical and horizontal lines is a defect or even an injustice. These students can be reassured. It will be shown in Chapter 6 that every line without exception has an equation of the form $px + qy + r = 0$. This includes our case $y = ax + b$, because we can rewrite this equation as $ax - y + b = 0$.

Exercises 3-10

1. Find the equation which defines the linear function f such that
   (a) $f(1) = 1$; $f(3) = 3$
   (b) $f(1) = 3$; $f(3) = 1$
   (c) $f(1) = 3$; $f(-3) = 4$
   (d) $f(7) = 0$; $f(8) = 42$.
2. What is the equation of the line which goes through
   (a) $(1,1)$ and $(3,3)$?
   (b) $(1,3)$ and $(3,1)$?
   (c) $(1,3)$ and $(-3,4)$?
   (d) $(7,0)$ and $(8,42)$?
3. Find the equations of two linear functions for which $f(1) = 2$. Try to describe the set of all such functions. What point do their graphs have in common?
4. Describe all linear functions f for which
   (a) $f(0) = 0$
   (b) $f(0) = 6$
   (c) $f(6) = 0$

   In each case try to interpret your answer geometrically.

3-11. Miscellaneous Problems.

1. Each of the following expressions suggests or defines a function. Describe its domain, its range and its rule.
   (a) The perimeter of a hexagon.
   (b) The length of the circumference of a circle depends on the length of its diameter.

(c) The pairings $(1,9)$, $(2,4)$, $(3,9)$.

(d) Hourly temperature readings.

(e) To each number x is assigned $x^2 + 4x - 2$.

(f) The pairings $(1,a)$, $(2,a)$, $(3,a)$.

(g) Assign to each real number x a number $x^2 + 3$.

(h) The distance an automobile travels depends on its speed and the elapsed time.

(i) To each odd positive integer assign -1 and to each even positive integer assign 1 .

(j) Distances to the nearest mile of each state capitol from Washington, D. C.

2. Given that f is the function which assigns $3x^2 - 5$ to each real number x .
   (a) What is the domain of f ?
   (b) What is the range of f ?
   (c) $f(0) = ?$
   (d) $f(-1) = ?$
   (e) $f(5) = ?$
   (f) $f(a) = ?$
   (g) $f(a - 1) = ?$
   (h) $f(\pi) = ?$

3. Given that f is the function whose domain is the set of all integers which assigns $+1$ to all non-negative integers and -1 to all negative integers.
   (a) What is the range of f ?
   (b) $f(-3) = ?$
   (c) $f(0) = ?$
   (d) $f(3) = ?$
   (e) $f(2 - 6) = ?$
   (f) $f(2) - f(6) = ?$
   (g) $f(4) + f(2) = ?$
   (h) $f(4 + 2) = ?$
   (i) $f(-6) = ?$
   (j) $f(-6) + 3 = ?$
   (k) $f(3 \cdot 6) = ?$
   (l) $3f(6) = ?$

4. Plot the graphs of the functions defined by the following equations.
   (a) $y = 5x - 4$
   (b) $y = -x^2 + 1$
   (c) $y = 3x^2 - 6$
   (d) $y = (x - 3)^2$
   (e) $y = 2x^3$

5. Let $f(x) = x^2 + 3$ and $g(x) = 2x + 5$.
   (a) $f(g(x)) = ?$
   (b) $g(f(x)) = ?$

6. Each of the following equations defines a linear function f . Find its inverse g and check that $f(g(x)) = g(f(x))$.
   (a) $y = x + 5$
   (b) $y = -2x - 1$
   (c) $y = -3x + 7$
   (d) $y = 5x - 6$

7. Find the linear function f such that
   (a) $f(3) = 5$, $f(5) = 3$
   (b) $f(1) = 0$, $f(-3) = 1$
   (c) $f(-2) = 3$, $f(3) = -2$
   (d) $f(0) = 5$, $f(5) = 2$

8. (a) If f is a constant function does

$$f(x + 1) = f(x) ?$$

   (b) If f is a function such that

$$f(x + 1) = f(x)$$

   must f be a constant function or a linear function?

9. (a) If f is a linear function does
   $f(x + 2) - f(x + 1) = f(x + 1) - f(x)$?
   (b) If f is a function such that
   $f(x + 2) - f(x + 1) = f(x + 1) - f(x)$
   must f be a constant or a linear function?

10. Let f be the function defined by $y = x^3 + 1$. Does f have an inverse? If so, what is the equation which defines the inverse of f ?

11. Let A be the set of real positive numbers and let f be the function with domain A defined by $y = \frac{1}{x + 1}$.
    (a) What is the range of f ?
    (b) What is the equation which defines g, the inverse of f ? What is the domain of g ? What is the range of g ?

12. Given that f is a function for which

$$f(x) = ax^2 + bx + c \quad , \quad a \neq 0 \; .$$

Prove that if

$$g(x) = f(x + 1) - f(x)$$

then g is a linear function.

13. (a) Find an equation which defines a linear function that is its own inverse.

    (b) Describe the set of all linear functions which are their own inverses.

Chapter 4

QUADRATIC FUNCTIONS AND EQUATIONS

4-1. Quadratic Functions.

Definition 4-1: Let a, b, c be any real numbers. Then if $a \neq 0$ we call the function defined by the equation

$$y = ax^2 + bx + c$$

a *quadratic function*.

We are going to study quadratic functions by examining a succession of special cases. We begin with the function defined by $y = x^2$, and then progress to the function defined by

$$y = ax^2,$$

$$\text{by} \quad y = a(x - k)^2,$$

$$\text{by} \quad y = a(x - k)^2 + p$$

and eventually arrive at the general case of the function defined by

$$y = ax^2 + bx + c.$$

In each case we shall try to see what the graph of the function looks like.

Exercises 4-1

Which of the following equations define a quadratic function?

1. $y = x^2$
2. $y = 2x$
3. $y = 2x^2$
4. $y = \frac{2}{x^2}$
5. $y = x^2 + 1$
6. $y = 2x + 1$
7. $y = x^2 + x$
8. $y = x(x - 1)$
9. $y = x(x - 1)(x - 2)$
10. $y = 2^x$

For what values of t do the following equations define a quadratic function?

11. $y = tx^2 + 3x + 4$
12. $y = x^2 + tx + 4$
13. $y = (t - 2)x^2 + 1$

14. $y = \frac{1}{t} x^2 + 2x + 3$

15. $y = x^t + 2x + 3$

Each of the following equations is equivalent to an equation of the form $y = ax^2 + bx + c$. For each find a,b and c.

16. $y = 3x^2$

17. $y = 3(x - 4)^2$

18. $y = 3(x - 4)^2 + 5$

19. $y = (x + 2)(x - 3)$

20. $y = (4x + 7)(3x - 2)$

4-2. The Function Defined by $y = x^2$.

The equation $y = x^2$ defines a function whose domain is the set of all real numbers. We recall some facts about real numbers to help us sketch the graph of this function. We saw in Chapter 1 that the equation $x^2 = k$ has no solution if $k < 0$, has one solution if $k = 0$, namely 0, and has two solutions if $k > 0$, namely $\sqrt{k}$ and $-\sqrt{k}$. We also know that if y_1 and y_2 are positive numbers, then the positive solution of $x^2 = y_1$ is less than the positive solution of $x^2 = y_2$ if and only if y_1 is less than y_2. If we use only these facts to sketch the graph of the function defined by $y = x^2$, we could obtain a graph which looks like Figure 4-2a. This graph has a single lowest point (0,0). For positive values of x, y increases indefinitely as x increases indefinitely. The graph is symmetric with respect to the y-axis. Actually the graph of the function defined by $y = x^2$ does not have the wobbly appearance of Figure 4-2a. It really looks like the curve shown in Figure 4-2b. We accept this fact now, without proof, on the understanding that the proof will be supplied

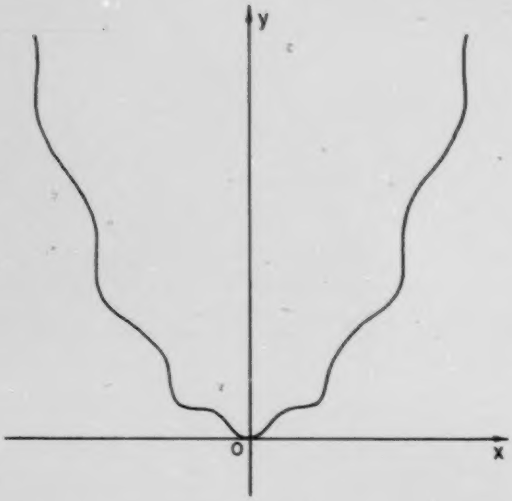


Figure 4-2a

later. The curve of Figure 4-2b is called a parabola, the point P is called its vertex and the line $x = 0$ is called its axis.

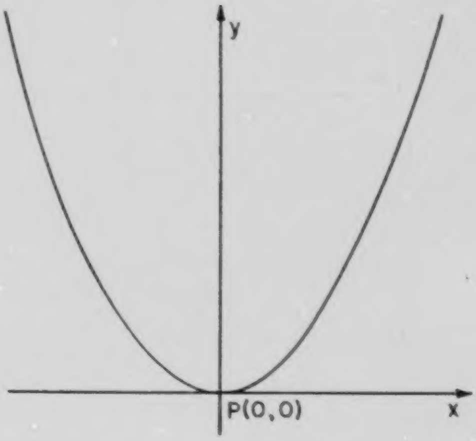


Figure 4-2b

Example 4-2a: Plot the graph of $y = x^2$.

Solution: Draw up a table of values.

x	-3	-2	-1	0	1	2	3
y	9	4	1	0	1	4	9

Plot these points and draw a smooth curve through them.

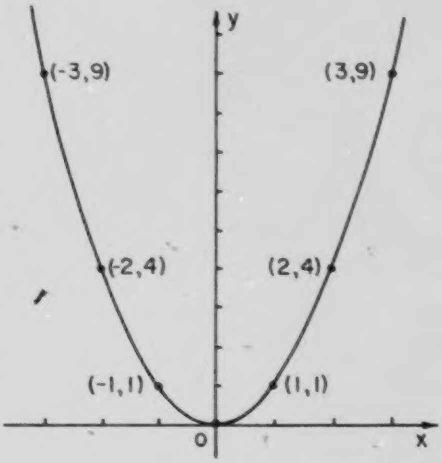


Figure 4-2c

Exercises 4-2

1. Plot the graph of $y = x^2$. For the following values of x, determine the corresponding values of y by calculation and also by measuring on the graph.

   (a) $x = \frac{3}{2}$

   (b) $x = -\frac{5}{2}$

   (c) $x = -\frac{5}{4}$

   (d) $x = \frac{4}{3}$

   (e) $x = -\frac{1}{2}$

2. Choose several points on the graph of $y = x^2$, measure their coordinates and check that these numbers satisfy the equation $y = x^2$.
3. What is the graph of $y = x^2$
   (a) if only points whose coordinates are integers are considered?
   (b) if only points whose coordinates are rational numbers are considered?

4-3. The Function Defined by $y = ax^2$.

For each value of a the equation $y = ax^2$ defines a function. These functions are best studied in two cases:

Case I: $a > 0$

(1) For $y < 0$, there are no values of x which satisfy $y = ax^2$.

(2) For $y = 0$, there is one value of x which satisfies $y = ax^2$ namely 0.

(3) For each $y > 0$, there are two values of x which satisfy $y = ax^2$, namely $\sqrt{\frac{y}{a}}$ and $-\sqrt{\frac{y}{a}}$.

(4) For any given x, as a increases, y increases.

Figure 4-3a shows graph of $y = ax^2$ for $a = \frac{1}{10}$, $\frac{1}{2}$, 1, 2, 5.

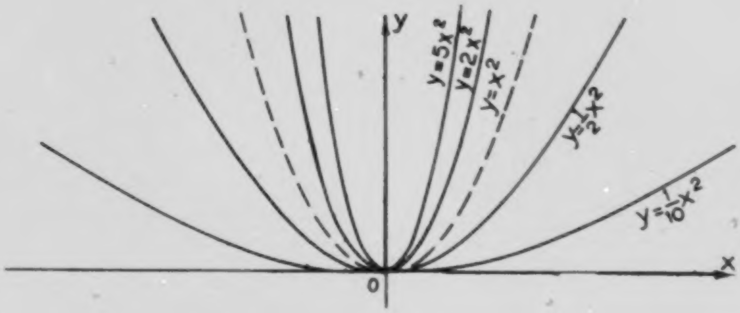


Figure 4-3a

Notice that all these curves have the same vertex (0,0), the same axis $x = 0$ and all open upward. Notice also that the smaller values of $|a|$ correspond to the "flatter" curves.

Case II: $a < 0$.

The graph of $y = ax^2$ with $a < 0$ can be obtained from the graph of the equation where $a > 0$ by a geometric construction. For instance, suppose we wished to draw the graph of $y = -4x^2$. First observe that a pair (x,y) satisfies the equation $y = 4x^2$ if and only if the pair $(x,-y)$ satisfies $y = -4x^2$. Next observe that (x,y) and $(x,-y)$ are symmetrical to each other with respect to the x-axis. Therefore to plot the graph of $y = -4x^2$, all we have to do is "reflect" the graph of $y = 4x^2$ in the x-axis.

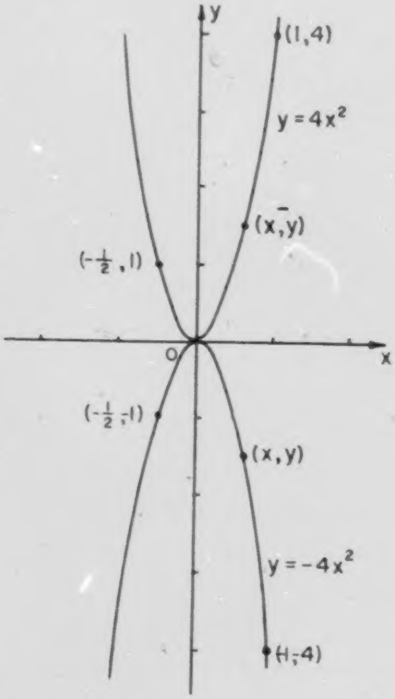


Figure 4-3b

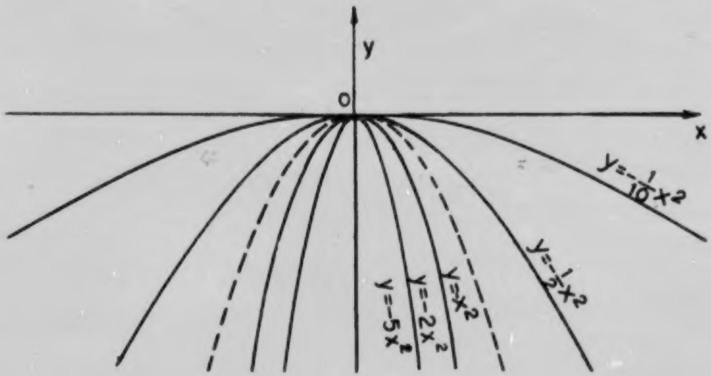


Figure 4-3c

Figure 4-3c shows the graph of $y = ax^2$ for $a = -5, -2, -1, -\frac{1}{2}$ $-\frac{1}{10}$.

Exercises 4-3

1. Plot the graph of each of the following equations:
   - (a) $y = 2x^2$
   - (b) $y = -2x^2$
   - (c) $y = \frac{1}{2} x^2$
   - (d) $y = -\frac{1}{2} x^2$
2. For each of the following determine a so that the graph of $y = ax^2$ contains the given point.
   - (a) (1,1)
   - (b) (1,2)
   - (c) (2,1)
   - (d) (1,-1)
   - (e) (-2,1)
   - (f) (-2,2)

3. For each of the following pairs of equations, given that (u,v) is on the graph of the first and that (u,w) is on the graph of the second, which of the following is correct? $v > w$, $v = w$, $v < w$.

(a) $y = 3x^2$
$y = -3x^2$

(b) $y = 3x^2$
$y = 4x^2$

(c) $y = -3x^2$
$y = -4x^2$

4-4. The Function Defined by $y = ax^2 + c$.

Let us now consider the graph of the function defined by the equation $y = ax^2 + c$. The figure 4-4a shows the graphs of four functions which are representative of this case; these are:

(1) $y = x^2 + 1$

(2) $y = x^2 + 2$

(3) $y = x^2 - 1$

(4) $y = x^2 - 2$.

So that you may compare the graphs of the new class of functions with that of the familiar $y = x^2$, the graph of the latter has been sketched in with a dashed line. By studying the figure you can see that the graph of $y = x^2 + 2$ is congruent to the graph of $y = x^2$, but that for the same x the ordinate of $y = x^2 + 2$ is two units more than the corresponding ordinate of $y = x^2$. Similarly for the same x the ordinate of $y = x^2 - 2$ is two units less than the corresponding ordinate of $y = x^2$. Thus the lowest point on the graph of $y = x^2 + 2$ is $(0,2)$ and the lowest point of $y = x^2 - 2$ is $(0,-2)$. Note that each of these graphs has a minimum point.

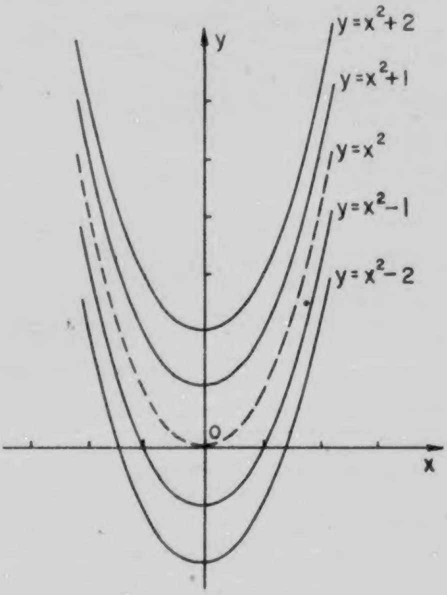


Figure 4-4a

Figure 4-4b shows the graph of $y = -x^2 + c$ for various values of c. Notice that in this case each of these graphs has a maximum point.

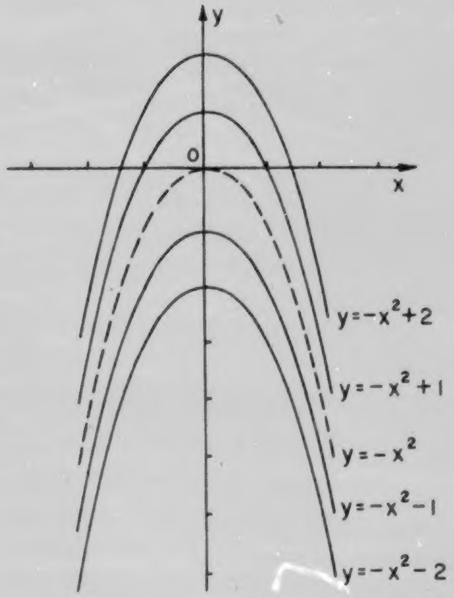


Figure 4-4b

The graphs of all functions defined by equations of the form $y = ax^2 + c$ are each symmetric with respect to the y-axis, regardless of the particular values of a and c. As before, the smaller values of $|a|$ give "flatter" curves.

We may summarize by saying that the graph of $y = ax^2 + c$ is congruent to the graph of $y = ax^2$, but has a position which is $|c|$ units up or down according as c is positive or negative. In each case the curve cuts the y-axis at $(0,c)$.

Exercises 4-4

1. Find the vertex and axis of the graph of each of the following equations.

   (a) $y = 5x^2 + 1$

   (b) $y = -5x^2 + 2$

1. (c) $y = \frac{1}{3} x^2 - 1$

   (d) $y = 3x^2 - \frac{1}{3}$

   (e) $y = - \frac{1}{10} x^2 + 6$

2. Sketch the graph of each of the equations in problem 1.

3. For each of the following pairs of equations, plot the graphs using a single set of coordinate axes.

   (a) $y = 2x^2 + 3$
   $y = 2x^2 - 3$

   (b) $y = \frac{1}{2} x^2 + 3$
   $y = \frac{1}{2} x^2 - 3$

   (c) $y = -2x^2 + 3$
   $y = -2x^2 - 3$

   (d) $y = -x^2 + 1$
   $y = +x^2 + 1$

   (e) $y = -2x^2 - 1$
   $y = 2x^2 - 1$

   (f) $y = -3x^2 + 1$
   $y = 3x^2 + 1$

4. Which of the functions in problem 1 have a minimum value and which have a maximum value? What are these values?

5. For each of the following pairs of equations, given that (u,v) is on the graph of the first equation and that (u,w) is on the graph of the second, which of the following is correct? $v > w$, $v = w$, $v < w$.

   (a) $y = 3x^2 - 4$
   $y = -3x^2 - 4$

   (b) $y = 3x^2 - 4$
   $y = 3x^2 + 6$

4-5. The Function Defined by $y = a(x - k)^2$.

In this section we study functions defined by equations of the form

$$y = a(x - k)^2$$

where a and k are non-zero constants. We proceed by considering several examples.

Example 4-5a: Make a table of values and plot the graph of $y = 2(x - 3)^2$.

x	...	1	2	3	4	5
$y = 2(x - 3)^2$	...	8	2	0	2	8

The axis of this curve is the line $x = 3$. Its vertex is the point $(3,0)$.

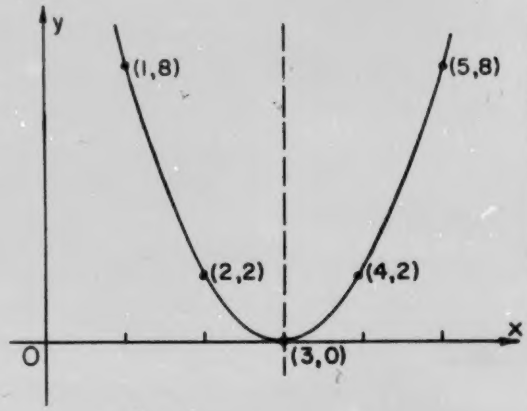


Figure 4-5a

Example 4-5b: Make a composite table of values for $y = 2x^2$ and $y = 2(x - 3)^2$ and plot the graphs of the two functions on the same set of axes.

x	...	-2	-1	0	1	2	3	4	5	...
$y = 2x^2$	...	8	2	0	2	8	18	...	...	...
$y = 2(x - 3)^2$	...	...	...	18	8	2	0	2	8	...

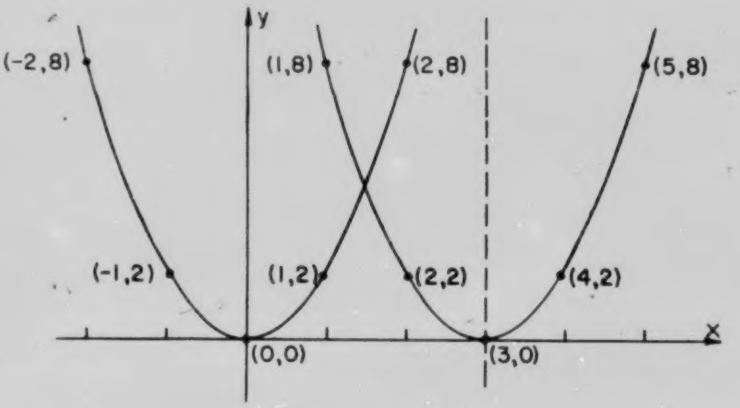


Figure 4-5b

The graph of $y = 2x^2$ is symmetric with respect to the line $x = 0$, and the graph of $y = 2(x - 3)^2$ is symmetric with respect to the line $x = 3$.

Summary of properties of the graph of $y = a(x - k)^2$.

1. The graph is congruent to the graph of $y = ax^2$, but has a position $|k|$ units to the right or left of the graph $y = ax^2$ according as $k > 0$ or $k < 0$.

2. If $a > 0$, the graph opens upward and has a lowest point $(k,0)$; if $a < 0$ the graph opens downward and has a highest point $(k,0)$.

3. The graph is symmetric with respect to the line $x = k$, and this line is called the axis of the graph.

Exercises 4-5

1. Find the vertex and the axis of the graph of each of the following equations.

   (a) $y = (x - 2)^2$ (b) $y = -2(x + 1)^2$

1. (c) $y = \frac{1}{2}(x - 1)^2$ (e) $y = 3(x - 2)^2$
   (d) $y = -\frac{1}{3}(x + 2)^2$ (f) $y = -5(x - 1)^2$
2. Sketch the graph of each of the equations in problem 1.
3. For each of the following pairs of equations, plot the graphs using a single set of coordinate axes.
   (a) $y = (x - 3)^2$ (d) $y = \frac{1}{2}(x - 1)^2$
   $y = -(x - 3)^2$ $y = \frac{1}{2}(x + 1)^2$
   (b) $y = -(x + 1)^2$ (e) $y = \frac{3}{4}(x - 1)^2$
   $y = -(x - 1)^2$ $y = -\frac{3}{4}(x - 1)^2$
   (c) $y = -2(x + 4)^2$ (f) $y = 2(x + \frac{1}{2})^2$
   $y = 2(x - 4)^2$ $y = -2(x + \frac{1}{2})^2$
4. Which of the graphs in problem 1 have a minimum value and which have a maximum value? What are these values?
5. For each of the following pairs of equations, given that (u,v) is on the graph of the first equation and that (u,w) is on the graph of the second determine the values of u for which $v < w$, $v = w$, $v > w$.
   (a) $y = 3(x - 4)^2$ (b) $y = 3(x - 4)^2$
   $y = -3(x - 4)^2$ $y = 3(x + 4)^2$

4-6. The Function Defined by $y = a(x - k)^2 + p$.

We know that the graph of $y = ax^2 + p$ has a position which is $|p|$ units up or down the graph of $y = ax^2$, and from the last section we know that the graph of $y = a(x - k)^2$ has a position that is $|k|$ units to the right or left of the graph $y = ax^2$. Hence, the graph of $y = a(x - k)^2 + p$ is congruent to the graph of $y = ax^2$ but is $|p|$ units up or down and $|k|$ units to the right or left of the graph of $y = ax^2$. The expressions "up" and "to the right" are associated with positive values of p and k, and "down" and "to the left" are associated with negative values.

Example 4-6a: Plot the graphs of $y = 2(x + 3)^2 + 1$, and $y = 2(x - 3)^2 + 1$ using a single set of axes.

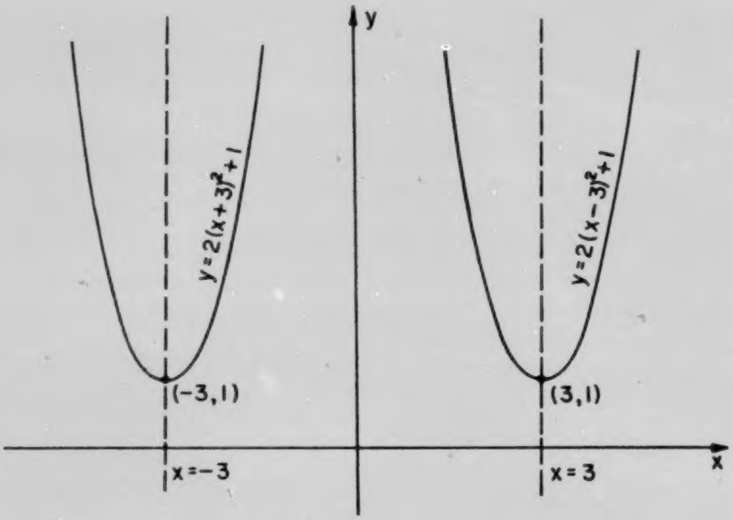


Figure 4-6a

The graph of $y = 2(x - 3)^2 + 1$ has a lowest point (3,1) and has the line $x = 3$ as its axis. The graph of $y = 2(x + 3)^2 + 1$ has a lowest point (-3,1) and has the line $x = -3$ as its axis. Notice that both curves open upward.

Summary of properties of graph of $y = a(x - k)^2 + p$.

1. If $a > 0$ the graph opens upward and the curve has a lowest point (k,p). If $a < 0$ the graph opens downward and has a highest point (k,p).
2. The graph has the line $x = k$ as its axis.

Exercises 4-6

1. Find the vertex and the axis of the graph of each of the following equations.

   (a) $y = 2(x - 3)^2 + 4$ (d) $y = -\frac{1}{2}(x - 1)^2 - 1$

   (b) $y = -2(x - 3)^2 + 4$ (e) $y = 3(x + 1)^2 + 2$

   (c) $y = (x + 3)^2$ (f) $y = \frac{1}{5}(x - 2)^2 - 3$

2. Sketch the graph of each of the equations in Problem 1.

3. For each of the following pairs of equations plot the graphs using a single set of coordinate axes.

   (a) $y = 2(x - 1)^2 + 3$ (d) $y = -2(x + 1)^2 + 3$
   $y = 2(x + 1)^2 + 3$ $y = 2(x - 1)^2 - 3$

   (b) $y = -2(x - 1)^2 - 3$ (e) $y = 3(x + 1)^2 + 2$
   $y = 2(x + 1)^2 + 3$ $y = (x + 1)^2 + 2$

   (c) $y = -2(x + 1)^2 + 3$ (f) $y = -3(x - 1)^2 + 2$
   $y = -2(x + 1)^2 - 3$ $y = (x - 1)^2 - 2$

4. Which of the exercises in problem 1 have a minimum value and which have a maximum value? What are these values?

5. For each of the following pairs of equations, given that (u,v) is on the graph of the first equation and (u,w) is on the graph of the second, determine the values of u for which $v < w$, $v = w$, $v > w$.

   (a) $y = 2(x - 3)^2 + 6$
   $y = 2(x - 3)^2 - 6$

   (b) $y = 2(x - 3)^2 + 6$
   $y = -2(x - 3)^2 - 6$

   (c) $y = 2(x - 3)^2 + 6$
   $y = 2(x + 3)^2 + 6$

4-7. The Function defined by $y = ax^2 + bx + c$.

We turn now to the general quadratic function defined by the equation $y = ax^2 + bx + c$, and reduce the study of this function to the special cases studied in the previous sections. We do this by performing a useful algebraic manipulation known as "completing the square".

Let us examine a few examples first. Consider the function defined by $y = 3x^2 - 6x + 3$. Since $3x^2 - 6x + 3 = 3(x^2 - 2x + 1) = 3(x - 1)^2$, our equation is of the form $y = a(x - k)^2$ and is covered in section 4-5. Suppose now we have a more complicated example, say $y = 3x^2 - 6x + 4$. This cannot be written in the form $y = a(x - k)^2$. However we can write $y = (3x^2 - 6x + 3) + 1$ and conclude that $y = 3(x - 1)^2 + 1$. Thus this second equation is of the type $y = a(x - k)^2 + p$ studied in section 4-6. In both examples we started with an expression $ax^2 + bx + c$ and ended with a new expression, equal to the original one, of the form $a(x - k)^2 + p$. These two examples are typical of what happens in general. Every expression $ax^2 + bx + c$ can be written in the form $a(x - k)^2 + p$ provided only that a is not zero. The following theorem states this fact and also shows how k and p can be found.

Theorem 4-7a. If a is not zero then
$$ax^2 + bx + c = a(x - k)^2 + p \text{ where } k = -\frac{b}{2a} \text{ and } p = \frac{4ac - b^2}{4a}$$

Proof:
$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{bx}{a}\right) + c \\ &= a\left(x^2 + \frac{bx}{a} + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} \end{aligned}$$

Note: This proof depends on a few algebraic maneuvers. In the first place the expression $\frac{b^2}{4a^2}$ was added to $x^2 + \frac{b}{a}x$ to obtain the square $x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = \left(x + \frac{b}{2a}\right)^2$. Notice also that adding $\frac{b^2}{4a^2}$ inside the parentheses having the multiplier a on the outside amounts to adding $a\frac{b^2}{4a^2}$, which is $\frac{b^2}{4a}$, to the whole expression. The expression $-\frac{b^2}{4a}$ was therefore added to the whole expression to be sure that the final expression was equal to the original.

Example 4-7a.
$$\begin{aligned} 3x^2 - 6x + 4 &= 3(x^2 - 2x) + 4 \\ &= 3(x^2 - 2x + 1) - 3 + 4 \\ &= 3(x - 1)^2 + 1 \end{aligned}$$

Here $a = 3$, $b = -6$, $c = 4$, $\frac{+b}{2a} = -1$, $\frac{4ac - b^2}{4a} = 1$. The graph of $y = 3x^2 - 6x + 4$ is shown in Figure 4-7a. Its vertex is the point whose coordinates are (1,1). Its axis is the line whose equation is $x = 1$. The graph does not go below the line whose equation is $y = 1$.

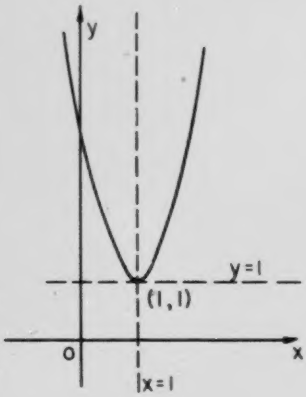


Figure 4-7a

Example 4-7b.
$$\begin{aligned} x^2 - 4x - 6 &= (x^2 - 4x) - 6 \\ &= (x^2 - 4x + 4) - 4 - 6 \\ &= (x - 2)^2 - 10 \end{aligned}$$

Here $a = 1$, $b = -4$, $c = -6$, $\frac{+b}{2a} = -2$, $\frac{4ac - b^2}{4a} = -10$. The graph of $y = x^2 - 4x - 6$ is shown in Figure 4-7b.

Its vertex is the point whose coordinates are (2,-10). Its axis is the line whose equation is $x = 2$. The graph does not go below the line whose equation is $y = -10$.

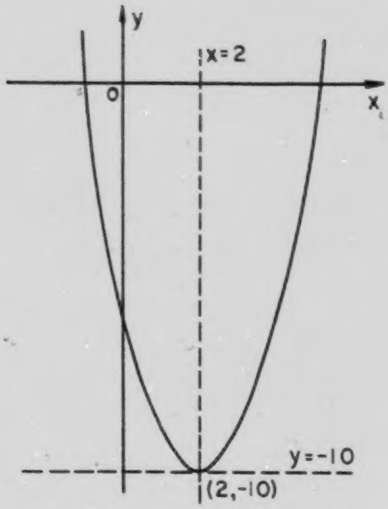


Figure 4-7b

Example 4-7c.

$$-6x^2 + 7x - 8 = -6(x^2 - \frac{7}{6}x) - 8$$

$$= -6(x^2 - \frac{7}{6}x + \frac{49}{144}) + \frac{49}{24} - 8$$

$$= -6(x - \frac{7}{12})^2 - \frac{143}{24}$$

Here $a = -6$, $b = 7$, $c = -8$,

$$\frac{b}{2a} = \frac{-7}{12}, \quad \frac{4ac - b^2}{4a} = \frac{-143}{24}$$

The graph of $y = -6x^2 + 7x - 8$ is shown in Figure 4-7c. Its vertex is the point whose coordinates are $(\frac{7}{12}, \frac{-143}{24})$. Its axis is the line whose equation is $x = \frac{7}{12}$. The graph does not go above the line whose equation is $y = \frac{-143}{24}$.

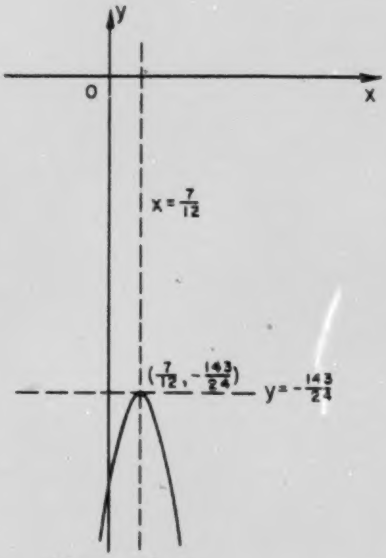


Figure 4-7c

Exercises 4-7

1. Transform the following equations to the form $y = a(x - k)^2 + p$, by completing the square

   (a) $y = x^2 - 4x$ (f) $y = x^2 - 144$

   (b) $y = 2x - x^2$ (g) $y = x^2 + 2x - 3$

   (c) $y = x^2 + 3$ (h) $y = 2x^2 + 8x - 5$

   (d) $y = 3x^2 + 5$ (i) $y = x^2 + 2x - 24$

   (e) $y = -x^2 + 6x + 7$ (j) $y = 10 + 5x - 5x^2$

2. Find the vertex and axis of the graph of each of the following equations:

   (a) $y = x^2 + 7x - 8$ (f) $y = x^2 - \frac{7}{2}x + 3$

   (b) $y = -x^2 - 11x - 31$ (g) $y = 5x^2 + 4x + 3$

   (c) $y = -2x^2 - x - 1$ (h) $y = -3x^2 + 2x - 2$

   (d) $y = 4x^2 + x - 3$ (i) $y = -5x^2 + 3x$

   (e) $y = -2x^2 - 5x - 1$ (j) $y = 2x^2 + 8$

3. Sketch the graph of each of the equations in problem 2.

4-8. Quadratic Functions having Prescribed Values.

Every quadratic function makes infinitely many pairings of one real number with another. It is reasonable to ask how many of these pairings can be prescribed arbitrarily. It turns out that the answer to this question is three. Let us state this fact more specifically. Let x_1, x_2, x_3 be any distinct real numbers and let y_1, y_2, y_3 be any three real numbers whatsoever. Then, if there is no linear function which pairs x_1 with y_1, x_2 with y_2 and x_3 with y_3, there is one and only one quadratic function which makes these pairings.

We are not in a position to prove this fact now because its proof requires solving systems of three equations in three unknowns and this topic is not discussed until Chapter 8. Let us look into an example anyway. Suppose we try to find a quadratic function which pairs 3 with 1, 9 with -1 and 6 with 2. We would look for a quadratic function defined by $y = ax^2 + bx + c$ such that

$$3 = a(1)^2 + b(1) + c = a + b + c$$

$$9 = a(-1)^2 + b(-1) + c = a - b + c$$

$$6 = a(2)^2 + b(2) + c = 4a + 2b + c.$$

We seek three numbers a, b, c which satisfy these equations. It can be checked that $a = 2$, $b = -3$ and $c = 4$ satisfy the equations and that the quadratic function defined by $y = 2x^2 - 3x + 4$ makes the given pairings. A method by which these numbers can be found is described in Chapter 8.

This question about the quadratic function also has a geometric version. It has to do with prescribing points to lie on a single parabola. It turns out that if any three points are given which do not lie on a line then they lie on the graph of some equation $y = ax^2 + bx + c$.

For instance if the points with coordinates (1,3), (-1,9) and (2,6) are given, then the graph of the equation $y = 2x^2 - 3x + 4$ contains these points. The diagram shows that these points do lie on the parabola and that they are not collinear.

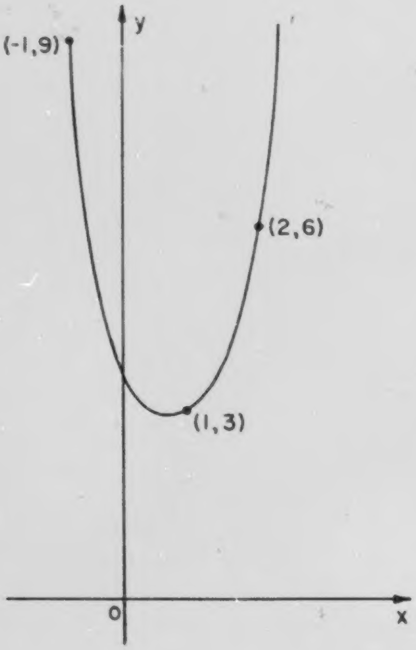


Figure 4-8a

Exercises 4-8

1. Find the quadratic function which pairs 0 with 0, 1 with 1 and +1 with -1.
2. Find the quadratic function whose graph passes through (0,0), (2,0) and (1,-1).
3. Find the quadratic function which pairs 0 with 0, 0 with 2 and -1 with 1.
4. Determine the number t so that the graph of the equation $y = ax^2 + bx + c$, $a \neq 0$, contains the points (0,0), (1,2) and (-1,t).

4-9. Equivalent Equations; the Equation $ax^2 + bx + c = 0$.

Definition 4-9a. Two equations are said to be equivalent if and only if they have the same solution set.

Example 4-9a. The equation $2x - 6 = 0$ and $x - 3 = 0$ are equivalent since the solution set of each is {3}.

There are several ways of manipulating an equation to obtain an equivalent equation. Some of these ways are

(1) addition of the same number to both members of an equation.

(2) multiplication of both members of an equation by the same non-zero number.

For instance the equation $2x = -6$ is obtainable from $2x + 6 = 0$ by adding -6 to both members, and these are equivalent equations. The equation $x = -3$ is obtainable from $2x = -6$ by multiplying both members by $\frac{1}{2}$ and these are equivalent equations.

We are going to continue our study of the equation $y = ax^2 + bx + c$. We have already seen that the function defined by this equation pairs certain values of y with 2 values of x, certain values of y with no values of x and one value of y with exactly one value of x. We are going to consider y as a given number and examine the solution set of the equation $y = ax^2 + bx + c$ regarded as an equation in x.

As a first step we simplify the problem by reducing it to the study of equations of the form $0 = ax^2 + bx + c$. We shall see shortly how useful this step is. We ought also to convince ourselves that no cases are lost by considering only this special case with $y = 0$. For instance the equation $17 = 3x^2 + 4x + 5$ is equivalent to the equation $0 = 3x^2 + 4x - 12$. More generally, if y, a, b, c are numbers, then the equation $y = ax^2 + bx + c$ is equivalent to the equation $0 = ax^2 + bx + c'$ where $c' = c - y$. This can be shown by adding $-y$ to both members of the first equation.

Exercises 4-9

Show that the following pairs of equations are equivalent.

1. $3x + 9 = 0, \quad x + 3 = 0$
2. $2x + 6 = 9, \quad x = \frac{3}{2}$
3. $x^2 + 9x + 10 = 0, \quad 2x^2 + 18x + 20 = 0$
4. $x^3 + 7x^2 + 3x + 9 = 0, \quad x^3 + 7x^2 + 3x + 12 = 3$
5. $\frac{x^2}{2} - 8 = 0, \quad x^2 = 16$
6. $17x + x^2 = 11, \quad x^2 + 17x - 11 = 0$
7. $x^2 + 7x + 3 = 20, \quad 0 = -17 + 7x + x^2$
8. $-3x^2 + 4x - 9 = 6 + x, \quad -3x^2 + 3x - 15 = 0$
9. $5x^2 - 15x = 0, \quad x^2 - 3x = 0$
10. $ax^2 + bx + c = 0, \quad -c - bx = ax^2$

Find a quadratic equation of the form $ax^2 + bx + c = 0$ equivalent to each of the following:

11. $x^2 + 20 = 8x + 5$
12. $x^2 + 3x = 2x + 6$
13. $x^2 + 49 = 14x$
14. $2x^2 + 3x + 7 = x^2 + 3x + 6$
15. $4x^2 + 8x = 5$

Test the following pairs of equations to see if they are equivalent:

16. $4x - 3 = 0, \quad x = \frac{3}{4}$

17. $x^2 = \frac{9}{16}$, $x = \frac{3}{4}$

18. $x^2 - a^2 = 0$, $x^2 + a^2 = 0$

19. $x^2 = 1$, $x = 1$

20. $x^2 = 0$, $x = 0$

21. $(x - 50)^{125} = 0$, $(x - 50)^{13} = 0$

4-10. Solution of $ax^2 + bx + c = 0$ by Completing the Square.

Definitions 4-10a. If a, b, c are any real numbers and if $a \neq 0$, we say that the equation $ax^2 + bx + c = 0$ is a quadratic equation. A root of the equation $ax^2 + bx + c = 0$ is any member of the solution set of this equation. To solve an equation means to find its solution set.

Note: Any root of an equation can be called "a solution" of that equation. When the words "the solution of an equation" are used they refer to the entire solution set of that equation.

Theorem 4-10a. The quadratic equation

$$ax^2 + bx + c = 0$$

is equivalent to the equation

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} .$$

Proof: In section 4-7 we showed that

$$ax^2 + bx + c = a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a}$$

therefore $ax^2 + bx + c = 0$ is equivalent to

$$a\left(x + \frac{b}{2a}\right)^2 + \frac{4ac - b^2}{4a} = 0.$$

Adding $\frac{b^2 - 4ac}{4a}$ to both sides gives the equivalent equation

$$a\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a} .$$

Multiplying both sides of the equation by the non-zero number $\frac{1}{a}$ gives the equivalent equation

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}$$

Example 4-10a: $3x^2 + 4x + 5 = 0$ is equivalent to each of the following:

$$3(x^2 + \frac{4}{3}x) + 5 = 0$$

$$3(x^2 + \frac{4}{3}x + (\frac{2}{3})^2) + 5 - \frac{4}{3} = 0$$

$$3(x + \frac{2}{3})^2 + \frac{11}{3} = 0$$

$$3(x + \frac{2}{3})^2 = -\frac{11}{3}$$

$$(x + \frac{2}{3})^2 = -\frac{11}{9}$$

Theorem 4-10b. The quadratic equation $ax^2 + bx + c = 0$ has

(1) No roots if $b^2 - 4ac < 0$

(2) One root if $b^2 - 4ac = 0$, namely $-\frac{b}{2a}$

(3) Two roots if $b^2 - 4ac > 0$, namely

$$\frac{-b - \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

Note: This theorem refers only to those roots which are real numbers. When the complex numbers are introduced in Chapter 5 a different version of this theorem will be presented.

Proof:. We know that our equation, $ax^2 + bx + c = 0$, is equivalent to

$$(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2} .$$

According to Theorem 1-10b this equation has a solution if and only if the right member is not negative. Since the denominator $4a^2$ of this right member is the square of the non-zero number $2a$ it is positive. It follows that the right member is negative if and only if its numerator $b^2 - 4ac$ is negative. Thus the equation $ax^2 + bx + c = 0$ has no solution if $b^2 - 4ac < 0$. If $b^2 - 4ac = 0$, then $x + \frac{b}{2a} = 0$ and $x = -\frac{b}{2a}$. If $b^2 - 4ac > 0$, then either

$$x + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{or}$$

$$x + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}}$$

In the first case

$$x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$$

and in the second

$$x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

Corollary: A quadratic equation has at most two roots.

Example 4-10b. Consider the equation $3x^2 + 4x + 5 = 0$. We have $a = 3$, $b = 4$, $c = 5$, $b^2 - 4ac = 4^2 - 4 \cdot 3 \cdot 5 = -44$. This equation therefore has no solution according to Theorem 4-10b. The equation was also treated in example 4-10a and shown to be equivalent to the equation

$$(x + \frac{2}{3})^2 = -\frac{11}{9}.$$

The fact that the right member of this equation is negative, and consequently the equation has no solution, illustrates the central idea of the proof of Theorem 4-10b.

Example 4-10c: Consider the equation

$$3x^2 + 6x + 3 = 0.$$

We have $a = 3$, $b = 6$, $c = 3$, $b^2 - 4ac = 6^2 - 4 \cdot 3 \cdot 3 = 0$. According to Theorem 4-10b this equation therefore has exactly one solution, namely $-\frac{6}{2 \cdot 3} = -1$. The equation is equivalent to $3(x^2 + 2x + 1) = 0$ or $(x + 1)^2 = 0$. This latter equation clearly has -1 as its only root.

Example 4-10d: Consider the equation

$$5x^2 + 5x - 30 = 0.$$

We have $a = 5$, $b = 5$, $c = -30$, $b^2 - 4ac = 5^2 - 4 \cdot 5(-30) = 625$. According to Theorem 4-10b this equation has the two solutions

$$\frac{-5 + \sqrt{625}}{10} \quad \text{and} \quad \frac{-5 - \sqrt{625}}{10}.$$

Since $\sqrt{625} = 25$, these numbers are $\frac{-5 + 25}{10}$ and $\frac{-5 - 25}{10}$, that is 2 and -3. Let us check to see if 2 and -3 are in fact roots of our equation. Substituting 2 for x, we have

$$5 \cdot 2^2 + 5 \cdot 2 - 30 = 20 + 10 - 30 = 0$$

Substituting -3 for x, we have

$$5 \cdot (-3)^2 + 5(-3) - 30 = 45 - 15 - 30 = 0$$

Therefore 2 and -3 are roots of the given equation.

Definition 4-10b. The discriminant of the quadratic equation $ax^2 + bx + c = 0$ is the number $b^2 - 4ac$.

Corollary to Theorem 4-10b: A quadratic equation has

(1) No solution if its discriminant is negative.

(2) Exactly one root if its discriminant is zero.

(3) Exactly two roots if its discriminant is positive.

Theorem 4-10b amounts to giving three procedures for dealing with quadratic equations. When complex numbers are introduced, a single formula will cover all the cases, namely

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

As long as we are dealing with real numbers, we can only use this formula for the case $b^2 - 4ac \geq 0$.

Exercises 4-10

Test the following quadratic equations to determine which has no solution, which has one solution and which has two solutions, by completing the square.

1. $x^2 - 5x + 6 = 0$
2. $2x^2 - 3x - 5 = 0$
3. $3x^2 + 2x + 4 = 0$
4. $2x^2 + 3x = 0$
5. $x^2 + x + 1 = 0$
6. $x^2 - 6x + 9 = 0$
7. $5x^2 = 2x - 1$
8. $25x^2 = -10x - 1$
9. $5x^2 = 3x - 2$
10. $2x^2 + 9x = 4$

Test the following quadratic equations to determine which has no solution, which has one solution and which has two solutions, by finding the value of the discriminant.

11. $x^2 - 2x + 4 = 0$
12. $3x^2 - 4x - 2 = 0$
13. $x^2 - 9x + \frac{81}{4} = 0$
14. $2x^2 - 3x - 8 = 0$
15. $15x^2 + 5x + 1 = 0$
16. $8x^2 = 2x + 4$
17. $5x^2 = 2x - 1$
18. $x^2 + 8x + 4 = 0$

Solve by completing the square

19. $x^2 - 6x = 7$
20. $x^2 - 8x = 180$
21. $x^2 - 4x - 45 = 0$
22. $x^2 - 2x + 7 = 150$
23. $x^2 + 45 = 14x$
24. $x^2 + 5x + 1 = 0$
25. $x^2 + 3x = 10$
26. $x^2 + \frac{3}{2}x + \frac{1}{2} = 0$
27. $y^2 + \frac{11}{3}y = \frac{4}{3}$
28. $6x^2 - 5x = -1$

Solve by using the quadratic formula:

29. $x^2 + 2x - 4 = 0$
30. $5y^2 + 9y + 4 = 0$
31. $3t^2 = 2t + 2$
32. $4y^2 = 3y + 2$
33. $x^2 + 7x - 1 = 0$
34. $x^2 + x - 1 = 0$
35. $7x^2 + 21x = 2$
36. $12x^2 + x - 1 = 0$
37. $2x^2 + 11x - 3 = 0$
38. $cx^2 + bx + a = 0$
39. $bx^2 + ax + c = 0$
40. $bx^2 + cx + a = 0$

Solve by any method:

41. $5x - 3x^2 = 0$
42. $2x^2 - x - 3 = 0$
43. $x^2 + x - 1 = 0$
44. $x^2 + 8x = 2$
45. $5x^2 - x - 3 = 0$
46. $2x^2 - 5x - 6 = 0$
47. $2x^2 + x - 3 = 0$
48. $25x^2 + 10x + 1 = 0$
49. $16x^2 = 8x + 9$
50. $3x^2 + 5x - 7 = 0$

4-11. Solution of Quadratic Equations by Factoring.

Theorem 4-11a. If a, r and s are real numbers and if $a \neq 0$, then $\{r,s\}$ is the solution set of the equation

$$a(x - r)(x - s) = 0.$$

Proof: First we show that r and s are roots of the equation. If we substitute r for x we obtain

$$a(r - r)(r - s) = a \cdot 0(r - s) = 0,$$

so r is a root of the equation. If we substitute s for x, we obtain

$$a(s - r)(s - s) = a(s - r)\cdot 0 = 0$$

so s is a root of the equation. There remains to show that no other number is a root of the equation. This follows from the theorem that if a product of real numbers is zero, one of the factors must be zero. If a number t is a root of the equation then $a(t - r)(t - s)$ must be zero. Since $a \neq 0$ it follows that either $t - r$ is zero or $t - s$ is zero. We conclude that if t is a root either $t = r$ or $t = s$.

Example 4-11a: Consider the equation

$$17(x - 18)(x + 19) = 0 .$$

According to Theorem 4-11a its solution set is $\{18,-19\}$. To see this it is helpful to notice that $x + 19 = x - (-19)$.

Example 4-11b: Consider the equation

$$17(x - 18)^2 = 0 .$$

This equation has one root, 18. According to Theorem 4-11a the solution set of this equation is the set $\{18,18\}$ which is a correct but somewhat unusual way of designating the set $\{18\}$. Thus the theorem is valid for equations with only one root.

Theorem 4-11a is occasionally useful in solving quadratic equations $ax^2 + bx + c = 0$, because it is sometimes possible to factor $ax^2 + bx + c$ into an expression of the form $a(x - r)(x - s)$.

Example 4-11c: Consider the quadratic equation

$$x^2 - 3x + 2 = 0 .$$

It is equivalent to the equation

$$(x - 2)(x - 1) = 0$$

and the solution set of the original equation is therefore $\{2,1\}$.

Example 4-11d: Consider the quadratic equation

$$21x^2 + 11x - 2 = 0.$$

It is equivalent to the equation

$$(7x - 1)(3x + 2) = 0.$$

This is equivalent to the equation

$$7(x - \tfrac{1}{7})3(x + \tfrac{2}{3}) = 21(x - \tfrac{1}{7})(x + \tfrac{2}{3}) = 0.$$

The solution set of the original equation is therefore $\{\frac{1}{7}, -\frac{2}{3}\}$. The equation

$$(7x - 1)(3x + 2) = 0$$

can also be solved directly by solving

$$7x - 1 = 0$$

$$\text{and} \quad 3x + 2 = 0.$$

Exercises 4-11

Solve the following equations by factoring:

1. $x^2 - 5x + 6 = 0$
2. $x^2 - 8x + 16 = 0$
3. $x^2 - 16 = 0$
4. $x^2 - 3x - 54 = 0$
5. $2x^2 - 5x + 3 = 0$
6. $2x^2 + x - 3 = 0$
7. $16x^2 - 25 = 0$
8. $33x^2 - 11x = 0$
9. $x^2 + 8x - 65 = 0$
10. $10x^2 + 29x - 21 = 0$
11. $15x^2 - 6 = x$
12. $31x = 6x^2 + 35$
13. $x + 2 = 15x^2$
14. $2x^2 + 5x = 12$
15. $5x + 4 = 6x^2$
16. $6 + 7x = 5x^2$
17. $3x^2 + 5x = 0$
18. $11x = 2 + 15x^2$
19. $9x^2 - 16 = 0$
20. $7x^2 - 5 = 2x$
21. $21x^2 + 40x - 21 = 0$
22. $21x^2 + 11x - 2 = 0$
23. $34x^2 + 17x = 0$
24. $18x^2 - 9x + 1 = 0$
25. $64x^2 - 16x + 1 = 0$
26. $x^2 - 2ax - 24a^2 = 0$
27. $x^2 - 3bx - 4b^2 = 0$
28. $x^2 - (a + b)x + ab = 0$
29. $x^2 + (a - b)x - ab = 0$
30. $t^2x^2 - (at + bt)x + ab = 0$

4-12. Some Properties of the Roots of a Quadratic Equation.

Theorem 4-12a. If r and s are any real numbers there is a quadratic equation whose solution set is $\{r,s\}$.

Proof: Since

$$(x - r)(x - s) = x^2 - (r + s)x + rs$$

the equation

$$(x - r)(x - s) = 0$$

is equivalent to the quadratic equation

$$x^2 - (r + s)x + rs = 0.$$

Since the solution set of the first equation is $\{r,s\}$ this is also the solution set of the second.

Note: If r and s are equal to each other this argument is still valid. The solution set is then $\{r\}$ which is the same as $\{r,r\}$.

Sometimes such a quadratic equation, with only one root, is said to have "two equal roots" or a "double root". This terminology is discussed later in Chapter 5.

Example 4-12a: Consider the set $\{14,11\}$.

Since $(x - 14)(x - 11) = x^2 - 25x + 154$ the given set is the solution set of the quadratic equation

$$x^2 - 25x + 154 = 0.$$

Example 4-12b: Consider the set $\{14,-11\}$.

Since $(x - 14)(x - (-11)) = (x - 14)(x + 11) = x^2 - 3x - 154$ the given set is the solution set of the quadratic equation

$$x^2 - 3x - 154 = 0.$$

Corollary: If $\{r,s\}$ is the solution set of the equation $x^2 + px + q = 0$, then $r + s = -p$

and $rs = q$.

Proof: We consider our equation to be of the type $ax^2 + bx + c = 0$ with $a = 1$, $b = p$ and $c = q$. It follows then from the quadratic formula that the solutions of our equation are

$$\frac{-p + \sqrt{p^2 - 4q}}{2}$$

and

$$\frac{-p - \sqrt{p^2 - 4q}}{2}$$

The sum of these numbers is

$$\frac{-p + \sqrt{p^2 - 4q} - p - \sqrt{p^2 - 4q}}{2}$$

which simplifies to $-p$. The product of these numbers is

$$\frac{p^2 + p\sqrt{p^2 - 4q} - p\sqrt{p^2 - 4q} - (p^2 - 4q)}{4}$$

which simplifies to q.

Theorem 4-12b: If the solution set of the quadratic equation $ax^2 + bx + c = 0$ is $\{r,s\}$ then

$$r + s = -\frac{b}{a} \text{ and } r \cdot s = \frac{c}{a}$$

Corollary: If the quadratic equation $ax^2 + bx + c = 0$ has roots r and s, then $ax^2 + bx + c = a(x - r)(x - s)$.

Example 4-12a: Let us consider the equation

$$x^2 - 3x + 2 = 0$$

which is an instance of the equation discussed in Theorem 4-11a with $p = -3$ and $q = +2$. Its solution set is $\{2,1\}$. According to Theorem 4-12a we should have

$$2 + 1 = -p = -(-3)$$
$$2 \cdot 1 = q = 2$$

and these statements are indeed correct.

Example 4-12c: Consider the equation $21x^2 + 11x - 2 = 0$. We saw in example 4-11d that its solution set is $\{\frac{1}{7}, -\frac{2}{3}\}$. According to Theorem 4-12b we should have $\frac{1}{7} + (-\frac{2}{3}) = -\frac{11}{21}$ and $\frac{1}{7}(-\frac{2}{3}) = -\frac{2}{21}$ and these statements are indeed correct.

Exercises 4-12

For each of the following form a quadratic equation whose solution set is the given set.

1. $\{5,6\}$
2. $\{3,-7\}$
3. $\{4\}$
4. $\{6,-6\}$
5. $\{0,0\}$
6. $\{4,-\frac{1}{3}\}$
7. $\{\frac{2}{5}, 0\}$
8. $\{\frac{2}{3}, -\frac{3}{4}\}$
9. $\{4 - \sqrt{5}, 4 + \sqrt{5}\}$
10. Find a quadratic equation whose roots are r and $\frac{1}{r}$.

Find the sum and product of the roots of the following equations if roots exist.

11. $x^2 - 13x + 40 = 0$
12. $x^2 + 5x - 50 = 0$
13. $2x^2 - 6x + 5 = 0$
14. $2x^2 - 6x = 0$
15. $7x^2 - 11x - 8 = 0$
16. $x^2 - (p + q) + pq = 0$
17. The roots of an equation are $2 + \sqrt{3}$ and $2 - \sqrt{3}$. Find the sum and product of these roots and write the equation.
18. Construct a quadratic equation with integral coefficients which has the roots $\frac{-4 + \sqrt{5}}{3}$ and $\frac{-4 - \sqrt{5}}{3}$.
19. For each of the following find all values of h such that the equation has one root:

    (a) $5x^2 + 3x + h = 0$

    (b) $5x^2 + hx + 3 = 0$

    (c) $hx^2 + 3x + 2 = 0$

*20. Show that if one root of the equation $ax^2 + bx + c = 0$ is twice the other, then $2b^2 = 9ac$.

4-13. Equations Transformable to Quadratic Equations.

We consider equations which are not quadratic equations but which can be transformed to quadratic form. Since we know how to solve quadratic equations any problem reducible to a quadratic equation can be considered to be solved, so the advantage of this procedure is clear. There is a disadvantage to our procedure also; the transformations we use in this section do not always give us equations which are equivalent to the ones we started with.

We shall deal with transformations which can produce three different effects -- they can enlarge the solution set, diminish it or leave it unchanged. Specifically we shall deal with those transformations for which

(1) the solution set of the derived equation is the same as the solution set of the original equation (equivalent equations)

(2) the solution set of the derived equation is a proper subset of the solution set of the original equation (some roots get "lost")

(3) the solution set of the original equation is a proper subset of the solution set of the derived equation (extraneous roots are introduced).

Thus we cannot assume in what follows that the solution set of the derived quadratic equation is the solution set of the original equation.

We do not have rules for dealing with this subject. Instead we shall deal with some typical examples.

Example 4-13a: Solve the equation

$$x - 5 + \frac{4}{x} = 0$$

Clearly, the given equation is not in the quadratic form. We multiply both members of the given equation by x, obtaining

$$x(x - 5 + \frac{4}{x}) = 0$$

or

$$x^2 - 5x + 4 = 0.$$

This equation is a quadratic equation whose roots are 1 and 4. Let us verify that these are also roots of the original equation.

$$1 - 5 + \frac{4}{1} = 0$$

$$4 - 5 + \frac{4}{4} = 0$$

In this example multiplying both members of the original equation by x produced an equation all of whose roots were roots of the original one. This does not always happen as the next example shows.

Example 4-13b: Solve the equation

$$x + \frac{3}{x - 2} = 5 + \frac{3}{x - 2}$$

If we multiply both members of the equation by $x - 2$ the transformed equation is

$$x(x - 2) + 3 = 5(x - 2) + 3$$

or

$$x^2 - 2x + 3 = 5x - 10 + 3.$$

This equation is equivalent to the quadratic equation

$$x^2 - 7x + 10 = 0 .$$

The roots of this quadratic equation are 5 and 2. Let us check these in the original equation: $5 + \frac{3}{5 - 2} = 5 + \frac{3}{5 - 2}$ or $6 = 6$. Thus 5 is a root of the original equation. On the other hand we cannot substitute 2 for x in the original equation because this would produce a zero denominator. Therefore 2 is not a root of the original equation. Therefore 2 is an extraneous root.

Exercises 4-13a

Solve each of the following equations. Check your answers by substituting in the original equation.

1. $x - \frac{4}{x} + 3 = 0$
2. $x - \frac{9}{x} = 0$
3. $x + \frac{1}{x} = -2$
4. $x + \frac{1}{x} = 2$
5. $x - 1 = \frac{1}{x - 1}$
6. $(x + 5) + \frac{18}{x + 5} - 9 = 0$
7. $x - \frac{35}{x + 2} = 0$
8. $x - 9 = \frac{72}{x - 8}$
9. $\frac{x^2 + 10}{x - 5} = \frac{7x}{x - 5}$
10. $x - \frac{9}{x} = x$

Example 4-13c: Solve the equation

$$\sqrt{3x + 4} - x = 0 .$$

We first add x to both members, obtaining the equation

$$\sqrt{3x + 4} = x.$$

If we square both members of this equation we obtain the equation

$$3x + 4 = x^2$$

or

$$x^2 - 3x - 4 = 0.$$

The roots of this quadratic equation are 4 and -1. We check to see if either of these numbers is a solution of the original equation.

$$\sqrt{16} - 4 = 0$$
$$\sqrt{1} + 1 = 2$$

This shows that one of the two roots of the derived quadratic equation, namely +4 is a root of the original equation and that the other root, namely -1, is not.

Exercises 4-13b

Solve the following equations:

1. $\sqrt{10x} = x$
2. $\sqrt{6x + 7} = x + 2$
3. $\sqrt{x - 3} = x - 5$
4. $\sqrt{3x + 1} + 11 = 3x$
5. $\sqrt{5x - 1} = \sqrt{x} + 1$
6. $\sqrt{8x + 5} = \sqrt{2x} + 2$
7. $\sqrt{5x + 6} + \sqrt{3x - 2} = 6$
8. $\sqrt{6x + 7} - \sqrt{3x + 3} = 1$
9. $\sqrt{4x - 3} = \sqrt{8x + 1} - 2$
10. $\sqrt{2x - 5} + x = 2$

Example 4-13d: Solve $x^4 - 3x^2 + 2 = 0$.

This is not a quadratic equation. Let us substitute t for x^2. Then the transformed equation

$$t^2 - 3t + 2 = 0$$

is a quadratic equation. The roots of this equation are 1 and 2. The roots of the original equation are found by solving the equations $t = x^2 = 1$ and $t = x^2 = 2$. We obtain the numbers 1, -1, $\sqrt{2}$, $-\sqrt{2}$. We find by substitution that these numbers are roots. The solution set of the original equation is therefore $\{1, -1, \sqrt{2}, -\sqrt{2}\}$.

Example 4-13e: Solve $(x^2 - 5x)^2 - 2(x^2 - 5x) - 24 = 0$. This is not a quadratic equation. If we substitute z for $x^2 - 5x$, then the transformed equation

$$z^2 - 2z - 24 = 0$$

is a quadratic equation. Its roots are 6 and -4. The roots of the original equation are found by solving the equations

$$z = x^2 - 5x = 6$$

and

$$z = x^2 - 5x = -4.$$

The solution set of the first of these is $\{6, -1\}$, the solution set of the second is $\{4, 1\}$. We find by substitution in the original equation that the numbers 6, -1, 4, 1 are roots of the original equation, so the solution set of the original equation is $\{6,-1,4,1\}$.

Exercises 4-13c

Solve the following equations by making suitable substitutions.

1. $x^4 - 4x^2 + 3 = 0$
2. $x^4 - 6x^2 + 8 = 0$
3. $x^4 - 13x^2 + 36 = 0$
4. $x^4 - 29x^2 + 100 = 0$
5. $(x^2 - 3x)^2 - 2(x^2 - 3x) - 8 = 0$
6. $(x^2 + 3x)^2 + 3(x^2 + 3x) + 2 = 0$
7. $\frac{1}{(x - 2)^2} + \frac{5}{(x - 2)} - 36 = 0$
8. $(\frac{x}{x - 1})^2 - 3(\frac{x}{x - 1}) + 2 = 0$

Exercises 4-13d

Solve the following equations.

1. $2 - \frac{1}{x} = x$
2. $(x - 5)^2 + 2(x - 5) - 8 = 0$
3. $\frac{3x + 5}{6x + 5} = x - 1$
4. $x^4 - 8x^2 - 9 = 0$
5. $x^4 - 26x^2 + 25 = 0$
6. $(x^2 - x)^2 - 14(x^2 - x) + 24 = 0$
7. $(x^2 + 1)^2 + 6(x^2 + 1) + 8 = 0$
8. $\frac{x - 1}{x^2 - 4x + 4} - \frac{1}{2} = \frac{3}{x - 2}$
9. $\sqrt{2x - 3} + 2\sqrt{3x - 2} = 5$
10. $\sqrt{2x - 5} = 2 + \sqrt{x - 2}$
11. $3(x^2 + 3x)^2 - 2(x^2 + 3x) - 5 = 0$
12. $3(\frac{1}{x} + 1)^2 + 5(\frac{1}{x} + 1) = 2$
13. $\sqrt{x + 2} + \sqrt{3 + 2x} = 2$

14. $\sqrt{2x + 3} - \sqrt{4x - 1} = \sqrt{6x - 2}$

15. $\sqrt{x^2 + 9} + \frac{15}{\sqrt{x^2 + 9}} = 8$

16. $x^2 + 5x - 5 = \frac{6}{x^2 + 5x}$

17. $\frac{2x - 7}{x^2 + 3x + 2} = \frac{3x - 7}{x + 1} - \frac{4x - 2}{x + 2}$

18. $\frac{3x}{x - 2} + \frac{1}{4 - x^2} = 2$

19. $\frac{7}{x + 5} - \frac{8}{x - 6} = \frac{3}{x - 1}$

20. $2x^2 + (a + 2b)x + ab = 0$

4-14. Quadratic Inequalities.

By a quadratic inequality we mean an inequality of one of the following kinds

(1) $ax^2 + bx + c > 0$

(2) $ax^2 + bx + c < 0$.

The solution set of such an inequality can be found by examining the graph of the equation $y = ax^2 + bx + c$. The portions of the graph which are above the x-axis give the values of x which are the solution set of $ax^2 + bx + c > 0$. The portions of the graph which are below the x-axis give values of x which are the solution set of $ax^2 + bx + c < 0$.

Here are the cases which can come up

Case I (Figure 4-14a)

The solution set of $ax^2 + bx + c < 0$ is $\{x:p < x < q\}$.

The solution set of $ax^2 + bx + c = 0$ is $\{p,q\}$.

The solution set of $ax^2 + bx + c > 0$ is $\{x:x < p \text{ or } x > q\}$.

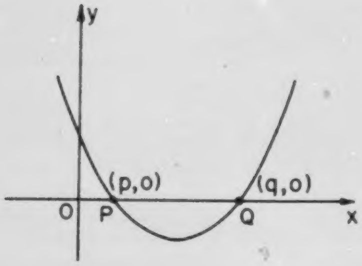


Figure 4-14a

Case II (Figure 4-14b)

The solution set of $ax^2 + bx + c < 0$ is the empty set.

The solution set of $ax^2 + bx + c = 0$ is $\{p\}$.

The solution set of $ax^2 + bx + c > 0$ is $\{x:x < p \text{ or } x > p\}$.

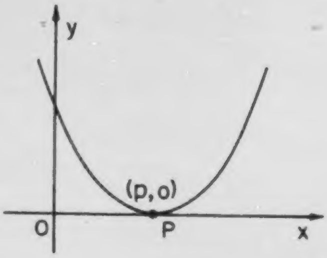


Figure 4-14b

Case III (Figure 4-14c)

The solution set of $ax^2 + bx + c < 0$ is the empty set.

The solution set of $ax^2 + bx + c = 0$ is the empty set.

The solution set of $ax^2 + bx + c > 0$ is the set of all real numbers.

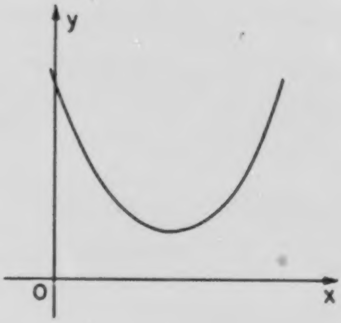


Figure 4-14c

Case IV (Figure 4-14d)

The solution set of $ax^2 + bx + c < 0$ is $\{x : x < p \text{ or } x > q\}$.

The solution set of $ax^2 + bx + c = 0$ is $\{p, q\}$.

The solution set of $ax^2 + bx + c > 0$ is $\{x : p < x < q\}$.

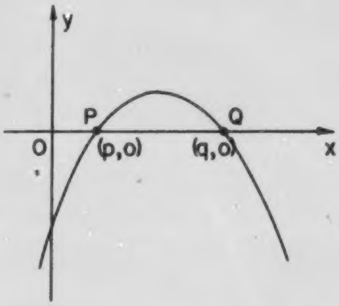


Figure 4-14d

Case V(Figure 4-14e)

The solution set of $ax^2 + bx + c < 0$ is $\{x : x < p \text{ or } x > p\}$.

The solution set of $ax^2 + bx + c = 0$ is $\{p\}$.

The solution set of $ax^2 + bx + c > 0$ is the empty set.

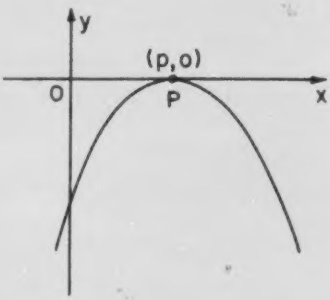


Figure 4-14e

Case VI (Figure 4-14f)

The solution set of $ax^2 + bx + c < 0$ is the set of all real numbers.
The solution set of $ax^2 + bx + c = 0$ is the empty set.
The solution set of $ax^2 + bx + c > 0$ is the empty set.

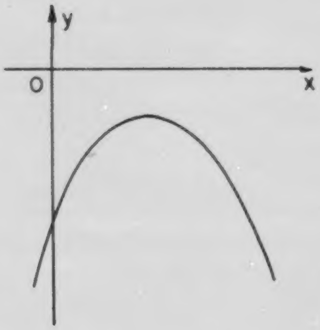


Figure 4-14f

Example 4-14a: Solve $x^2 - 5x + 4 > 0$.

Solution: First draw the graph of $y = x^2 - 5x + 4$. The solution set of $x^2 - 5x + 4 > 0$ is the set $\{x : x < 1 \text{ or } x > 4\}$. We arrive at this answer by determining the values of x for which the graph is above the x-axis.

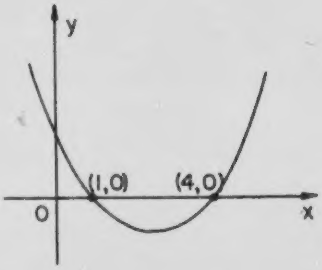


Figure 4-14g

Example 4-14b: Solve $-x^2 + 2x - 1 < 0$. The solution set of $-x^2 + 2x - 1 < 0$ is the set $\{x : x < 1 \text{ or } x > 1\}$. We arrive at this answer by determining the values of x for which the graph is below the x-axis.

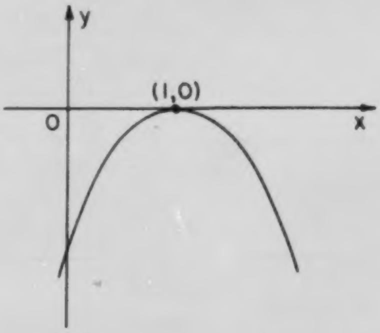


Figure 4-14h

Example 4-14c: Solve $-x^2 + 2x > 0$. Solution: First draw the graph of $y = -x^2 + 2x$. The solution set of $-x^2 + 2x > 0$ is the set $\{x : 0 < x < 2\}$. We arrive at this answer by determining the values of x for which the graph is above the x-axis.

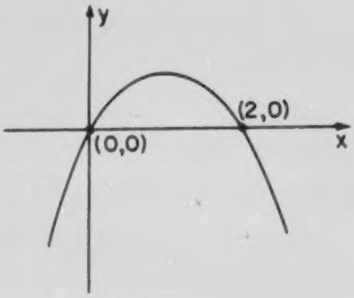


Figure 4-14i

Exercises 4-14

Find the solution set of each of the following inequalities.

1. $x^2 - 4x + 3 < 0$
2. $x^2 + 5x + 4 > 0$
3. $x^2 + x - 6 > 0$
4. $2x^2 + 4x + 5 < 0$
5. $x^2 - 16 < 0$
6. $x^2 - 6x + 9 < 0$

7. $x^2 - 6x + 8 < 0$
8. $-x^2 + 2x + 3 < 0$
9. $5x < 2 - 3x^2$
10. $3(x + 1) < 5x^2$
11. $6(-x^2 + 1) > 13x$
12. $-x^2 - 4x - 5 < 0$
13. $4x^2 + 1 > 4x$
14. $-x^2 + x > 0$
15. $2x - 1 > x - x^2$
16. Determine values of h for which each of the following equations has no roots, 1 root, 2 roots.
    (a) $x^2 + hx + 9 = 0$
    (b) $x^2 + hx + 9h = 0$

4-15. Applications.

Mathematics sometimes is divided into two parts, "pure" and "applied". The "pure" part is concerned with the logical analysis of mathematical objects, such as numbers and points; the "applied" part is concerned with using this knowledge to obtain information about other kinds of objects, such as speeds and places. For instance the statement $5 \cdot 52 = 260$ is an example of "pure" mathematics. It can be applied to solve the problem "how many cards are there in five decks of cards each consisting of fifty-two cards"? In this section we shall study a few problems arising outside the world of mathematics which can be formulated and solved as quadratic equations.

The fact that quadratic equations can have two roots sometimes introduces a slight complication. It can happen that the original problem has only one solution and that the auxiliary quadratic equation has two. Then common sense must be brought in to select the right root. For instance if the original problem is about the number of grains of sand on a beach, then any negative root of the auxiliary quadratic equation is surely not the right one.

Example 4-15a. On a river which flows at the rate of 3 miles an hour, a motorboat can go 12 miles downstream and 12 miles back in 2 hours and 8 minutes. What is rate of the boat in still water?

Solution: We are asked to find the rate of the boat in still water. We denote this unknown number by x. Then the boat travels downstream at the rate of $x + 3$ miles an hour. The number of hours it takes to go downstream is $\frac{12}{x+3}$. The number of hours it takes to return upstream is $\frac{12}{x-3}$. The number of hours for the entire trip is their sum

$$\frac{12}{x+3} + \frac{12}{x-3} .$$

Since we know that the total time is $2 + \frac{8}{60}$ hours we can express our problem mathematically by the equation

$$\frac{12}{x+3} + \frac{12}{x-3} = 2 + \frac{8}{60} = \frac{32}{15}$$

This is an equation which can be transformed to a quadratic by multiplying both members by $(x + 3)(x - 3)$. The transformed equation is

$$12(x - 3) + 12(x + 3) = \frac{32}{15}(x + 3)(x - 3)$$

or

$$24x = \frac{32}{15}(x^2 - 9).$$

This is equivalent to the quadratic equation

$$4x^2 - 45x - 36 = 0,$$

whose roots are 12 and $-\frac{3}{4}$. The number 12 is a possible solution of the original problem, the negative number $-\frac{3}{4}$ is not. We check to see if 12 satisfies the original equation by substituting 12 for x. We obtain

$$\begin{aligned}\frac{12}{12+3} + \frac{12}{12-3} &= \frac{4}{5} + \frac{4}{3}\\ &= \frac{32}{15}\end{aligned}$$

and conclude that the boat must travel 12 miles per hour in still water.

Example 4-15b: Many of the buildings of ancient Greece incorporated the proportions of the "golden rectangle". This figure is a rectangle ABCD having the property that if points P and Q are chosen on its longer sides so that APQD is a square then rectangle BCQP and rectangle ABCD are similar. Suppose it is required to find the base CD of such a rectangle if its height AD is to be 10 feet.

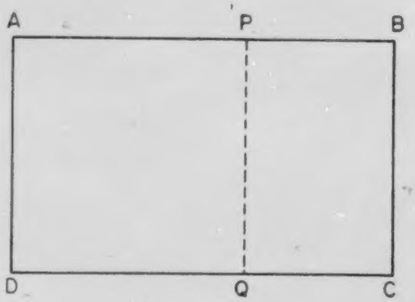


Figure 4-15

Solution: In the figure $AD = AP = PQ = DQ = 10$. We seek CD. We can express the geometric conditions concerning the similarity of the rectangles ABCD and BCQP by the algebraic equation $\frac{AB}{AD} = \frac{BC}{QC}$. If we denote CD by x, then $AB = x$ and $QC = x - 10$. The equation becomes

$$\frac{x}{10} = \frac{10}{x - 10}.$$

This can be transformed to

$$x(x - 10) = 100$$

which is equivalent to

$$x^2 - 10x - 100 = 0.$$

We solve this equation by use of the quadratic formula. Its two roots are

$$\frac{10 + \sqrt{500}}{2} \text{ and } \frac{10 - \sqrt{500}}{2}.$$

The second of these numbers is negative, and so cannot be the required length. We conclude that if the height of a "golden rectangle" is 10 feet, then the length of the base is $\frac{10 + \sqrt{500}}{2}$ feet. This is approximately 16 feet.

Exercises 4-15

1. The perimeter of a rectangle is 20 feet, its area is 21 square feet. Find its length and width.
2. A picture which is 9 inches wide by 12 inches long is surrounded by a frame. The area of the frame alone is

162 square inches. Find the width of the frame.

3. Find two consecutive positive integers whose cubes differ by 1261.

4. Assume that an object thrown vertically downward from the top of a cliff 2400 feet above a lake falls according to the law $s = 80t + 16t^2$ where s is the distance in feet that the object falls during the first t seconds.

   (a) How long does it take for the object to fall 224 feet from the top of the cliff?

   (b) How long does it take until the object strikes the surface of the lake?

   (c) Find the distance the object falls during the 8th and 10th seconds.

5. The edges of two cubes differ by 2 inches, and their volumes differ by 728 cubic inches. Find the dimensions of each.

6. A grocer sold oranges at a dollar a bag and raised the price per dozen by 10 cents by reducing the number of oranges in a bag by 4. Find

   (a) the original number of oranges in the bag;

   (b) the original price per dozen.

7. An engine pulls a train 140 miles. Then a second engine whose average speed is 5 miles per hour faster than the first engine takes over and pulls the train 200 miles. The total time required for the 340 miles is 9 hours. Find the average speed of each engine.

8. The square root of 3 less than twice a given number is 1 more than the square root of 2 more than the number. Find the number.

9. Find the dimensions of a rectangle if the diagonal is 2 more than the longer side, which in turn is 2 more than the shorter side.

10. Prove that there is no real number such that the sum of it and its reciprocal is 1.

11. Is there a rectangle with a perimeter of 66 inches, and an area of 260 square inches?

12. The sum of the squares of two consecutive integers is 113. What are these integers?

13. John and Mark leave St. Paul at the same time. John flies north while Mark flies east. Mark flies 100 miles an hour faster than John. At the end of an hour they are 500 miles apart. At what average speed did each travel?

14. The length of a rectangle is four times its width. Its area equals that of a square whose perimeter is 14 inches less than the perimeter of the rectangle. Find the dimensions of the rectangle.

*15. It is desired to make a rectangular pen to hold livestock. 100 yards of wire fencing materials are available. What dimensions will make the inclosed area a maximum? (Hint: Sketch graph.)

4-16. Miscellaneous Problems.

Plot the graphs of each of the following pairs of quadratic functions using a single set of axes for each pair. In each case specify the vertex and axis.

1. $y = x^2$
   $y = x^2 + 1$
2. $y = x^2 + 2$
   $y = (x - 2)^2$
3. $y = (x - 3)^2$
   $y = -(x - 3)^2$
4. $y = -2(x - 1)^2$
   $y = -2(x - 1)^2 + 3$
5. $y = -(x + 3)^2 - 1$
   $y = (x + 3)^2 + 1$
6. $y = x^2 - 4x + 4$
   $y = x^2 + 4x + 4$
7. $y = x^2 - 4x + 3$
   $y = x^2 + 4x + 3$
8. $y = -2x^2 - 4x - 2$
   $y = 2x^2 - 4x + 2$
9. $y = x^2$
   $y = x^2 + 3x$
10. $y = (x + 1)^2$
    $y = 2x^2 - 3x + 1$

Test the following quadratic equations to determine which has no solution, which has one solution and which has two solutions by finding the value of the discriminant. Also, find the sum and the product of the roots of each equation if roots exist.

11. $x^2 + x + 1 = 0$

12. $4x^2 + 12x + 9 = 0$

13. $t^2 + 2t - 2 = 0$
14. $2y^2 + 3y + 5 = 0$
15. $5t^2 - 3t - 4 = 0$
16. $x^2 = 18 + 7x$
17. $10x^2 = 3 + 13x$
18. $x^2 - 2x + 5 = 0$
19. $x^2 + 8x + 6 = 0$
20. $x^2 + 7x = 0$

Find the solution set of each of the following equations:

21. $2y^2 + y = 6$
22. $12t^2 + 31t - 15 = 0$
23. $2x^2 + 15x + 27 = 0$
24. $x^2 + 6x + 4 = 0$
25. $2x(x + 2) + 3 = 0$
26. $x^2 + \frac{3}{2}x + \frac{1}{2} = 0$
27. $t^2 + 5t + 1 = 0$
28. $4x^2 = 3x + 2$
29. $2y^2 + 11y - 3 = 0$
30. $36 + 36x + 9x^2 = 0$
31. $x^2 + 2x = 9$
32. $x^2 - 5x - 3 = 0$
33. $7x^2 - 10x + 5 = 0$
34. $2x^2 + 4x - 7 = 0$
35. $6x^2 - x - 3 = 0$
36. $6x - x^2 = 0$
37. $24x^2 - 86x - 15 = 0$
38. $2w^2 = 8w - 7$
39. $x - 5 = 3x^2$
40. $6 + 2x - x^2 = 0$

For each of the following equations determine k so that it has exactly one root.

41. $9x^2 + 30x + k = 0$
42. $kx^2 - 6x = 4$
43. $2x^2 + 8x + k = 0$
44. $9x^2 - 8kx = -4$
45. $kx^2 - kx + 1 = 0$

Form quadratic equations whose solution set is each of the following:

46. $\{3, -2\}$
47. $\{5, -5\}$
48. $\{2 + \sqrt{2}, 2 - \sqrt{2}\}$
49. $\{\frac{1}{4}, \frac{3}{2}\}$
50. $\{\frac{1}{3}, 3\}$

Find the solution set of the following equations:

51. $x - \sqrt{5x + 9} - 1 = 0$
52. $\sqrt{x^2 + 3} + \frac{4}{\sqrt{x^2 + 3}} = 4$
53. $\sqrt{3x - 5} + \sqrt{2x + 3} + 1 = 0$

54. $2x^4 - 17x^2 - 9 = 0$

55. $\frac{2x^2}{x + 1} + 1 = \frac{2}{x + 1}$

56. $(x^2 - 3x + 1)^2 - 4(x^2 - 3x + 1) - 5 = 0$

57. $\frac{x + 7}{x - 1} - \frac{2}{x - 7} = \frac{12x}{(x - 1)(x - 7)}$

58. $(a + b)^2(1 - x)x = ab$

59. $\sqrt{x + 4} + \sqrt{x - 1} = \sqrt{x - 4}$

60. $3x^4 - 4x^2 - 7 = 0$

Find the solution set of each of the following quadratic inequalities:

61. $x^2 - 4 < 0$

62. $x^2 - x - 2 < 0$

63. $2x^2 + 5x > 12$

64. $3x^2 + 14x - 5 < 0$

65. $2x^2 - 3x > 8$

Solve:

66. If 3 times the square of a certain number is decreased by 9 times the number, the result is 120. Find the number.

67. The length of a rectangle is 6 more than twice the width. Its diagonal is 39. Find length and width.

68. If a number is increased by 72, its positive square root is increased by 4. Find the number.

69. If the sum of two positive numbers is 50 what are the numbers if their product is to be a maximum? (Hint: Sketch graph.)

70. For what values of k does the equation $x^2 + 2kx + 9 = 0$ have no real roots?

71. What is the range of the function defined by the equation $y = 3x^2 - 6x + 5$?

72. Given the quadratic equation $kx^2 - 8x + 3 = 0$, find the value of k so that

(a) the solution set consists of one element.

(b) 3 is in the solution set.

73. Find the values of k for which the equation $kx^2 - 2x + 3 = 0$ has two distinct real roots.

74. For what values of r and s is $\{r,s\}$ the solution set of $x^2 + (r - 1)x + 2s = 0$?

75. The segment $\overline{AB}$ is 20 inches long. The point C is chosen on it so that AC is the mean proportional between CB and AB. Find AC.

Chapter 5

COMPLEX NUMBER SYSTEMS

5-1. Introduction.

In Chapter 4 we considered equations of the form

$$ax^2 + bx + c = 0, \tag{5-1a}$$

where a, b, c are real numbers, $a \neq 0$. We developed a method for solving such equations and found that the results depend in a very essential way on the value of the discriminant, $b^2 - 4ac$. If $b^2 - 4ac > 0$, the equation has two real solutions; if $b^2 - 4ac = 0$, the equation has one real solution; if $b^2 - 4ac < 0$, the equation has no real solution.

The time has come, it appears, to ask once more whether we can extend our number system to include numbers of such a character that every quadratic equation with real coefficients has a solution regardless of the value of its discriminant. It is the task of this chapter to make such an extension of the system of real numbers. Actually we shall find that the system we derive for this purpose is a richer one than we bargain for: It gives us the solutions not only of all quadratic equations with real coefficients, but also of all polynomial equations of whatever degree with real coefficients. Even this does not quite describe the richness of the system we derive, but it is too soon to tell the whole story. Let it suffice to say that no further extensions will be necessary for the purposes of ordinary algebra.

The simplest example of a quadratic equation with a negative discriminant is the equation

$$x^2 + 1 = 0. \tag{5-1b}$$

If this equation is written in the form (5-1a) we have a = 1, b = 0, c = 1, and the discriminant is

$$b^2 - 4ac = -4,$$

so that we know from Chapter 4 that it has no real solutions. We can see this without evaluating the discriminant. Since the square of each real number is non-negative, we have $x^2 \geq 0$ for any real number x. Thus, if x is real, $x^2 + 1 \geq 0 + 1 = 1 > 0$, so that no real number is a solution of equation (5-1b).

To start we will look for a number system in which Equation (5-1b) has a solution. It will turn out, in Section 5-5, that in this system every quadratic equation with real coefficients has a solution. Perhaps if you look again at the method of solving the quadratic equation in Section 4-10 you can now see why this should be so.

Before undertaking our extension of the system of real numbers, let us recall the procedure followed in Chapter 1 each time we extended a number system. We assumed that a new system could be constructed which would: (1) have all the algebraic properties of the old system; (2) include all the numbers of the old system, in such a way that the new and the old algebraic operations, when applied to numbers of the old system, would be the same; (3) contain new numbers of the kind we need. We then discovered the rules for operating with the new numbers as logical consequences of the properties we assumed.

Proceeding in the same way we now seek a new number system which contains the system of real numbers with all its familiar properties and also contains a number satisfying $x^2 + 1 = 0$, Equation (5-1b). We shall designate the system by the letter C and call it the system of complex numbers. Following are the specific properties we require of C:

Property C-1

(i) Two operations, addition (+) and multiplication (·) are defined in C. (It is to be understood that the result of an operation defined in a system is a number in the system, but when we wish to emphasize this fact we will say that the system is closed with respect to the operation.)

(ii) Addition is associative and commutative.

(iii) C possesses one and only one additive identity.

(iv) Each element of C has one and only one additive inverse.

(v) Multiplication is associative and commutative.

(vi) C possesses one and only one multiplicative identity.

(vii) Each element of C, other than the additive identity, has one and only one multiplicative inverse.

(viii) Multiplication is distributive with respect to addition.

Property C-2

(i) Every real number is a member of C.

(ii) The sum of two real numbers in C is the same as their sum in the real number system.

(iii) The product of two real numbers in C is the same as their product in the real number system.

(iv) The additive identity in C is the number 0 of the reals.

(v) The multiplicative identity in C is the number 1 of the reals.

Property C-3

The set C contains a special element i which has the property

$$i \cdot i = i^2 = -1.$$

We call the special element i the imaginary unit.

5-2. Complex Numbers.

In Section 5-1 we stated a problem: To find a number system — that is, a set of elements and the operations of addition and multiplication defined for the set — having properties C-1, C-2 and C-3. Now we try to solve this problem. Let us first try to identify the set of elements.

Property C-3 implies that C contains at least one member not in the set of real numbers because the square of no real number is negative. By C-1, C is closed under the operations of addition and multiplication, so that if a and b are real numbers, the product bi is in C since b and i are, and it follows that $a + bi$ is in C since a and bi are. We see, then, that all numbers of the form

$$a + bi, \text{ where } a \text{ and } b \text{ are real,}$$

are included in C. The number i and every real number can be written in this form. We have $i = 0 + 1 \cdot i$. If a is any real number $a = a + 0 \cdot i$, since $0 \cdot i = 0$. (The statement that the product of 0 and any number is 0 can be proved for numbers in C exactly as it was for integers in Chapter 1.)

Now, however, if we add and multiply numbers of this form, take their additive and multiplicative inverses, add and multiply again, and so on, it would seem that we should encounter more and more numbers of the system not of this form. This is not so! The sum and product, additive and multiplicative inverses of numbers which can be written in the form $a + bi$, a and b real, can be written in the same form. We have not proved this, but after we complete our discussion of operations with these numbers you will see how such a proof can be constructed.

The results we have stated imply that if there is any system which solves our problem, then there is a simplest - - that is, smallest possible — system which solves the problem. This is the system with the following property.

Property C-4 Each element of C can be written in the form $a + bi$, where a and b are real numbers.

We add C-4 to our list of basic properties; thus the system C which has Properties C-1, C-2, C-3 and C-4 is the system of complex numbers.

Historical Note. The adjectives "complex", "imaginary" -- and, by contrast, "real" -- which are standard terms sanctioned by years of use, serve to illustrate the "controversial" nature of our four fundamental properties. As recently as a hundred years ago many mathematicians believed that C-1, C-2, C-3 and C-4 contradicted one another, that is to say, that there could be no system with all these properties. The proof that this list of properties is just as respectable as that characterizing the "real" numbers was achieved through the work of the nineteenth century mathematicians, Argand, Cauchy and Gauss. (Such a proof is outlined in Section *5-10.) Our continued use of the classical adjectives serves to remind us of the old controversy and of the work of the men who resolved it.

Exercises 5-2

1. For each of the following pairs of number systems state a property of the first which is not possessed by the second:
   (a) integers, natural numbers
   (b) rational numbers, integers
   (c) real numbers, rational numbers
   (d) complex numbers, real numbers.
2. The following equations have solutions in the system of real numbers if a, b, and c are real numbers. For each equation name the smallest number system in which the equation has a solution in the system if a, b, and c are in the system.
   (a) $a + x = b$
   (b) $ax = b \quad a \neq 0$
   (c) $ax + b = c \quad a \neq 0$
3. Write each of the following complex numbers in the form $a + bi$ where a and b are real numbers.
   (a) 1 (c) -1 (e) 3 (g) i^2.
   (b) 0 (d) i (f) $2i$
4. For each of the following pairs of number systems state a property of the first which is not possessed by the second.
   (a) natural numbers, integers
   (b) real numbers, complex numbers.

*5. Let S be the set of all real numbers which can be written in the form $a + b\sqrt{2}$, where a and b are rational numbers. Show that
   (a) S is not the set of all real numbers.
   (Hint: Show that $\sqrt{3}$ is not in S.)

*5. (b) S is closed with respect to real addition and multiplication.

(c) the additive and multiplicative inverses of a number in S are also in S.

(d) S, with real addition and multiplication as operations, has all the properties listed in Property C-1.

(e) S is the smallest part of the real number system which has properties C-1, contains the rational numbers, and contains $\sqrt{2}$.

5-3. Addition, Multiplication and Subtraction.

We now take up the task of deducing rules for calculating with the complex numbers. The remainder of this section is devoted to theorems which give formulas for the sum, product and difference of two complex numbers. We postpone the discussion of division until Section 5-4.

Theorem 5-3a. $(a + bi) + (c + di) = (a + c) + (b + d)i$.

Proof: We suppose that $a + bi$ and $c + di$ are any two given complex numbers. Consider the expression

$$(a + bi) + (c + di).$$

Property C-1 assures us that addition in C is associative and commutative; therefore,

$$(a + bi) + (c + di) = (a + c) + (bi + di).$$

But Property C-1 also asserts that the distributive law holds, so $bi + di = (b + d)i$. Hence

$$(a + bi) + (c + di) = (a + c) + (b + d)i,$$

which we were required to prove.

Theorem 5-3b. $(a + bi)\cdot(c + di) = (ac - bd) + (ad + bc)i$.

Proof: Given complex numbers $a + bi$ and $c + di$, we consider the expression

$$(a + bi)(c + di).$$

Using the distributive law once, we obtain

$$(a + bi)(c + di) = a(c + di) + bi(c + di).$$

Applying the distributive law again, and using the commutative property of multiplication, we have

$$(a + bi)(c + di) = ac + adi + bci + bdi^2.$$

But $i^2 = -1$, so we can write

$$(a + bi)(c + di) = ac + adi + bci - bd.$$

Using the commutative property of addition and once again making use of the distributive law, we obtain

$$(a + bi)(c + di) = ac - bd + (ad + bc)i.$$

This completes the proof.

Example 5-3a. Express the sum of $2 + 3i$ and $5 + 2i$ in the form $a + bi$, where a and b are real numbers.

Solution: $(2 + 3i) + (5 + 2i) = (2 + 5) + (3 + 2)i = 7 + 5i.$

Example 5-3b. Express the product of $2 + 3i$ and $5 + 2i$ in the form $a + bi$, where a and b are real numbers.

Solution:
$$\begin{aligned}(2 + 3i)(5 + 2i) &= 2(5) - 3(2) + [2(2) + 3(5)]i\\ &= 10 - 6 + (4 + 15)i\\ &= 4 + 19i.\end{aligned}$$

Example 5-3c. Express the product of i, $2i$ and $1 - i$ in the form $a + bi$, a and b real.

Solution: $i \cdot 2i \cdot (1 - i) = -2(1 - i) = -2 + 2i.$

Now we consider subtraction. As in Chapter 1, we denote the additive inverse of z by $-z$, so that by definition

$$z + (-z) = 0. \tag{5-3a}$$

Also, just as in Chapter 1, we define $z_2 - z_1$ to be the solution z of the equation

$$z_1 + z = z_2, \tag{5-3b}$$

where z_1, z_2 are given. (We leave as an exercise the proof that Equation (5-3b) cannot be satisfied by more than one complex number z.) It is easy to see that $z_2 + (-z_1)$ is a solution of

Equation (5-3b).

$$z_1 + [z_2 + (-z_1)] = z_1 + [(-z_1) + z_2] = [z_1 + (-z_1)] + z_2$$
$$= 0 + z_2 = z_2.$$

We have therefore proved

$$(5\text{-}3c) \qquad z_2 - z_1 = z_2 + (-z_1).$$

Our problem now is to find $-z$ when $z = a + bi$ is given. Let $-z = x + yi$, where x and y are real. Since

$$z + (-z) = 0$$

we get

$$(a + bi) + (x + yi) = 0.$$

By the theorem on addition (Theorem 5-3a) this becomes

$$(a + x) + (b + y)i = 0 = 0 + 0\cdot i$$

and this equation will be satisfied if

$$a + x = 0, \qquad b + y = 0,$$

that is, if $x = -a$ and $y = -b$. Thus $(-a) + (-b)i$ is an additive inverse of $a + bi$, and since the inverse is unique we have proved the following:

Theorem 5-3c. If $a + bi$ is a complex number (a and b real), then its additive inverse is

$$-(a + bi) = -a + (-b)i.$$

We can now summarize our discussion of subtraction in a theorem.

Theorem 5-3d. $(a + bi) - (c + di) = (a - c) + (b - d)i.$

Proof: Using Formula (5-3c), Theorem 5-3a and Theorem 5-3c we have

$$\begin{aligned}(a + bi) - (c + di) &= (a + bi) + [-(c + di)]\\ &= (a + bi) + [(-c) + (-d)i]\\ &= [a + (-c)] + [b + (-d)]i\\ &= (a - c) + (b - d)i.\end{aligned}$$

Exercises 5-3

1. Express each of the following sums in the form $a + bi$, where a and b are real numbers:
   - (a) $(1 + 4i) + (3 + 5i)$
   - (b) $(2 + 6i) + (2 - 6i)$
   - (c) $(3 + 5i) + 2i$
   - (d) $4 + (\pi + \pi i)$
   - (e) $(\sqrt{2} + 3i) + (2i + 1)$
   - (f) $(-1 + 5i) + 2i$
   - (g) $8 + i$
   - (h) $3 + (7i - 3)$
   - (i) $(5 + 3i) + (7 + 2i) + (3 - 4i)$
   - (j) $(3 + 2i) + (\sqrt{2} + 7i) + \sqrt{3}\, i$.
2. Add a complex number to each of the following to make the sum a real number. Can this be done in more than one way?
   - (a) $2 - 5i$
   - (b) $x - yi$
   - (c) $\sqrt{2} - \sqrt{3}\, i$
   - (d) $-5i$
3. Express each of the following products in the form $a + bi$, where a and b are real numbers:
   - (a) $(2 + 3i)(4 + 7i)$
   - (b) $(2 - 3i)(6 + 4i)$
   - (c) $(3 - i)(1 + 2i)$
   - (d) $i(3 + 5i)$
   - (e) $2i(\sqrt{2} - i)$
   - (f) $(8 + \sqrt{2}\, i)(1 + \sqrt{3}\, i)$
   - (g) $(3 + 4i)(3 + 4i)$
   - (h) $(1 + i)(1 - i)$
   - (j) $6i \cdot 3i$
   - (k) $7i(-2i)(1 - 6i)$
   - (l) $(4 - 2i)(3 - 2i)(5i)$
   - (m) $(4 - 3i)^2(2 - 5i)$
   - (n) $(2 + 3i)(3 - 2i)(6 - 4i)$
   - (o) $(c + di)(x + yi)$
   - (p) $(x - y)(x + yi)$
4. Find the additive inverses of each of the following complex numbers and express them in the form $a + bi$, where a and b are real numbers:
   - (a) 3
   - (b) i
   - (c) $1 + i$
   - (d) $2 + 3i$
   - (e) $5 - 4i$
   - (f) $-4 - 3i$
   - (g) $a - bi$
   - (h) $x + yi$

5. Express each of the following differences in the form $a + bi$, where a and b are real numbers:
   (a) $(7 + 11i) - (2 + 3i)$
   (b) $(5 - 6i) - (7 - 8i)$
   (c) $(3 + 5i) - (3 - 5i)$
   (d) $i - (1 + i)$
   (e) $(\sqrt{3} + i) - (2 + \sqrt{2}\, i)$
   (f) $\sqrt{4} - (1 - i)$
   (g) $\pi - \pi i$
   (h) $(2 + 3i) - (2 - 3i)$
   (i) $(1 - i) - 2i$
6. Express the following powers of i in the form $a + bi$, where a and b are real numbers.
   (a) i^3
   (b) i^4
   (c) i^9
   (d) i^{15}
   (e) i^{4n+1}, n is a natural number
   (f) i^{79}
7. State a general rule for determining the n-th power of i where n is a natural number. Explain why the rule works.
8. Express each of the following quantities in the form $a + bi$, where a and b are real numbers.
   (a) $i^3 + i^4$
   (b) i^{4n+3}, n is a natural number
   (c) $3i + 4i(5 - i)(5 + i)$
   (d) $7i[(2 - 3i) + (4i + 10)]$
   (e) $i[(3i + 6) - (2i + 7)]$
   (f) $3(3 + 2i) + (6 + 8i) - 2(2 - 3i)$
   (g) $(b + c - ai)(a + c - bi)(a + b - ci)$, where a, b, c are real
   (h) $\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)^3$
   (i) $i(i - 1)(i - 2)(i - 3)$
9. Show by substitution that $\frac{3}{4} + \frac{\sqrt{7}}{4} i$ is a solution of the equation $2z^2 - 3z + 2 = 0$.

5-4. Standard Form of Complex Numbers.

Property C-4 asserts that each member of C can be expressed in the form $a + bi$, where a and b are real numbers. Our next theorem states that this representation is unique: given any complex number z, there is only one pair of real numbers a,b such that $z = a + bi$.

Theorem 5-4. If a, b, c, d are real numbers, then $a + bi = c + di$ if and only if $a = c$ and $b = d$.

Proof: The "if" part of the statement, "$a + bi = c + di$ if $a = c$ and $b = d$" is clear, since addition and multiplication have unique results. We have to prove the "only if" part: $a + bi = c + di$ only if $a = c$ and $b = d$, that is, if $a + bi = c + di$ then $a = c$ and $b = d$.

Suppose, accordingly that a, b, c, d are real numbers and that

$$a + bi = c + di.$$

Then by the theorem on subtraction (Theorem 5-3d),

$$(a - c) + (b - d)i = 0,$$

and

$$a - c = -(b - d)i.$$

We have to show that $a = c$ and $b = d$, or what is the same, that $a - c = 0$ and $b - d = 0$. Now if $b - d$ were not zero we could write

$$\frac{a - c}{b - d} = -i,$$

or

$$-\left(\frac{a - c}{b - d}\right) = i.$$

But this would imply that i is a real number since a, b, c, d are real numbers and the difference and quotient of real numbers are real. Since we know that i is not a real number we conclude that $b - d = 0$. But if $b - d = 0$, then $-(b - d)i = 0$, and since $(a - c) = -(b - d)i$, it follows that $a - c = 0$. This completes the proof.

Example 5-4a. Find all pairs of complex numbers x,y for which $2x + 3yi = 6 + 3i$.

Solution: One solution of the problem is $x = 3$, $y = 1$. If the problem had required that x and y be real then by the preceding theorem this would be the only solution. However, since we permit x and y to be complex, the preceding theorem is not directly applicable, and the equation may have other solutions; $x = 3 + 3i$, $y = -1$ is a solution, for example.

We can use Theorem 5-4 to find *all* complex solutions of this equation. Let $x = a + bi$, $y = c + di$ where a, b, c, d are real. Substituting in

$$2x + 3yi = 6 + 3i$$

we get

$$2(a + bi) + 3(c + di)i = 6 + 3i,$$

or

$$(2a - 3d) + (2b + 3c)i = 6 + 3i.$$

Since the expressions in parentheses in the last equation are real, it follows from the preceding theorem that the equation holds if and only if

$$2a - 3d = 6, \qquad 2b + 3c = 3;$$

or

$$c = \frac{3 - 2b}{3}, \quad d = \frac{2a - 6}{3}.$$

Here a and b may be assigned values arbitrarily. Thus, all the solutions of the equation are given by

$$x = a + bi, \qquad y = \frac{3 - 2b}{3} + \frac{2a - b}{3}\, i,$$

where a and b are any real numbers.

The representation of a complex number z as

$$z = a + bi,$$

where a and b are real numbers, is called the *standard form* of z. Note that z is real if and only if $b = 0$. (Why?) We therefore call a the *real part* of $a + bi$. The real number b is called the *imaginary part* of $a + bi$. Thus we can say that a complex number is real if and only if its imaginary part is zero. A complex number $a + bi$ in which $a = 0$ is called a *pure imaginary number*. Thus a complex number is a pure imaginary number if and only if its real part is zero. DO NOT CONFUSE the imaginary part b of the complex number $a + bi$ with the pure imaginary number bi. Both the real and imaginary parts of $a + bi$ are real numbers: they are the real numbers a and b, respectively. Usually a complex number which is not real is called imaginary.

Examples 5-4b

	z	Real part of z	Imaginary part of z	Standard form of z
1.	0	0	0	$0 + 0i$
2.	$2 + i$	2	1	$2 + 1i$
3.	$1 - i$	1	-1	$1 + (-1)i$
4.	i	0	1	$0 + 1i$
5.	i^2	-1	0	$-1 + 0i$

In these examples, only 0 and i^2 are real numbers; only 0 and i are pure imaginary numbers; $2 + i$, $1 - i$ and i are imaginary numbers.

Exercises 5-4

1. Find the real and imaginary parts of each of the following complex numbers:
    - (a) $(1 + i)^2$
    - (b) $1 + i^2$
    - (c) i^5
    - (d) $5 - i$
    - (e) $2x + 3i$
    - (f) $a - 2i$
    - (g) $(\sqrt{2} - i)^2$
    - (h) $(-1 + i\sqrt{3})^2$
    - (i) $(4 + i) - 7$
    - (j) $-2i^2$
    - (k) $3i$
    - (l) $2i + 1$
2. What real numbers must be added to each of the following complex numbers to make the sum a pure imaginary number? Can this be done in more than one way?
    - (a) $3 + 2i$
    - (b) $-4i$
    - (c) $5 - 2i$
    - (d) $5 - \sqrt{2}\, i$
3. Use Theorem 5-4a to find real values for x and y that satisfy the following equations:
    - (a) $x - yi = 3 + 6i$
    - (b) $2x + yi = 6$
    - (c) $x - 5yi = 20i$
    - (d) $8x + 3yi = 4 - 9i$
    - (e) $2x + 3yi - 4 = 5x - yi + 8i$
    - (f) $x - y + (x + y)i = 2 + 6i$
    - (g) $(1 + x) + i(2 - y) = 3 - 4i$
    - (h) $x + yi = 1 + i^2$
    - (i) $y^2 i^2 = i(1 - x^2)$
    - (j) $(x + i)^2 = y$

4. Express each of the following complex numbers in standard form:
   - (a) $3 + 2i + 5 + i$
   - (b) $(3 - 2i) - (5 - 2i)$
   - (c) $3i(4 - 2i)$
   - (d) $6 + 5i - (2 - 3i)$
   - (e) $(3 - 2i)(5 - 2i)$
   - (f) $(4 - i)(3 - 2i)$
   - (g) $(1 - i)(2 + 3i)(4 + 2i)$
   - (h) $(a + b - ci)(a + b + ci)$, where a,b,c are real numbers
   - (i) $(x + yi)^3$, where x and y are real numbers.

5. Suppose $z = x + yi$, where x and y are real numbers, and $z^2 = 8 + 6i$. Solve for x and y.

*6. Suppose, for the sake of this exercise, that a and b are complex numbers. Show that $a + bi = 0$ and $a - bi = 0$ if and only if $a = 0$ and $b = 0$. Show also that the underlined word can be replaced by "or" only when we also assume that a and b are real numbers.

*7. Show that if z_1 is any non-real complex number, every complex number z can be expressed in one and only one way in the form $z = a + bz_1$, where a and b are real numbers.

5-5. Division.

We have learned to add, multiply and subtract complex numbers. We now consider division.
According to Property Ç-1 every complex number other than 0 has one and only one multiplicative inverse. As in Chapter 1 we denote the multiplicative inverse of z by $\frac{1}{z}$, so that by definition

(5-5a) $$z \cdot \frac{1}{z} = 1.$$

Also, just as in Chapter 1, we define $\frac{z_2}{z_1}$ to be the solution z of the equation

(5-5b) $$z_1 \cdot z = z_2$$

when this solution exists. (We leave as an exercise the proof that equation 5-5b cannot be satisfied by more than one complex number z.) It is easy to see that if $z_1 \neq 0$, Equation 5-5b has the solution $z_2(\frac{1}{z_1})$:

$$z_1\cdot[z_2(\frac{1}{z_1})] = z_1[(\frac{1}{z_1})z_2] = [z_1(\frac{1}{z_1})]z_2 = 1\cdot z_2 = z_2.$$

We have therefore proved

(5-5c) $$\frac{z_2}{z_1} = z_2\cdot\frac{1}{z_1} \qquad z_1 \neq 0.$$

Our task now is to find the standard form of $\frac{1}{z}$ when $z = a + bi$ is given in standard form. Let us begin by considering a numerical example.

Example 5-5a. If $z = 2 + 3i$ find its multiplicative inverse $\frac{1}{z}$ in standard form.

Solution. We seek a number $x + yi$ (x and y real) satisfying

$$(2 + 3i)(x + yi) = 1.$$

If we multiply the factors on the left using the theorem on multiplication (Theorem 5-3b) we may write

$$(2x - 3y) + (3x + 2y)i = 1 + 0i.$$

Hence, from the theorem on standard form (Theorem 5-4),

$$2x - 3y = 1,$$
$$3x + 2y = 0.$$

Eliminating y, we have

$$(4 + 9)x = 2.$$

Hence

$$x = \frac{2}{13}, \quad y = \frac{-3}{13};$$

and

$$x + yi = \frac{2}{13} + (-\frac{3}{13})i.$$

Now we can verify by substitution that

$$\frac{1}{2 + 3i} = \frac{2}{13} + (-\frac{3}{13})i.$$

We treat the general case in exactly the same way. Suppose $a + bi$, in standard form, is a non-zero complex number. Recall that this means that at least one of the two real numbers a, b is not 0. If there is a complex number $x + yi$, x and y being real numbers which satisfy the equation

(5-5d) $$(a + bi)(x + yi) = 1,$$

then by completing the multiplication in the left member we get

$$(ax - by) + (bx + ay)i = 1.$$

From the theorem on standard form (Theorem 5-4), this equation will be satisfied if and only if

$$\begin{aligned} &ax - by = 1, \\ (5\text{-}5e)\qquad &bx + ay = 0. \end{aligned}$$

Thus our problem is reduced to that of solving two linear equations with real cofficients for the real unknowns x and y. We solve these equations by elimination. To eliminate y, multiply the first equation by a, the second by b, and add. We get

$$(a^2 + b^2)x = a.$$

Our assumption that $a + bi \neq 0$, i.e., that at least one of the real numbers a, b is not zero, tells us that $a^2 + b^2 \neq 0$. Hence we can write

$$x = \frac{a}{a^2 + b^2} .$$

In the same way, we eliminate x from Equations (5-5e). Multiplying the first equation by b, the second by a, and subtracting the first from the second, we get

$$(a^2 + b^2)y = -b.$$

As before, $a^2 + b^2 \neq 0$, so

$$y = \frac{-b}{a^2 + b^2} .$$

Now by substitution we can verify that

$$\frac{a}{a^2 + b^2} + \left(\frac{-b}{a^2 + b^2}\right)i$$

is a solution of Equation (5-5d) so that it is the unique multiplicative inverse of $a + bi$. We state our conclusion as a theorem.

Theorem 5-5. If $a + bi$ is a non-zero complex number in standard form, then its multiplicative inverse is

$$\frac{1}{a + bi} = \frac{a}{a^2 + b^2} + \left(\frac{-b}{a^2 + b^2}\right)i.$$

Now we can combine the results of this section to obtain a formula for the quotient of any two complex numbers when the denominator is not 0. We could state the result as a theorem, but the statement would be cumbersome. It is better to remember a procedure which we indicate by an example.

Example 5-5b. Find the quotient $\frac{8 + 5i}{2 + 3i}$ and express the answer in standard form.

Solution: By Formula 5-5c,

$$\frac{8 + 5i}{2 + 3i} = (8 + 5i)(\frac{1}{2 + 3i}).$$

By Theorem 5-5,

$$\frac{1}{2 + 3i} = \frac{2}{13} + \frac{-3}{13} i.$$

Combining these two equations and using the theorem on multiplication (Theorem 5-4b) we obtain

$$\frac{8 + 5i}{2 + 3i} = (8 + 5i)(\frac{2}{13} + \frac{-3}{13} i)$$

$$= \frac{31}{13} + (- \frac{14}{13})i$$

as the quotient in standard form.

The following relations involving division of complex numbers can be proved on the basis of Property C-1 just as they were in the real case.

(5-5f) $z_1 z_2 = 0$ if and only if $z_1 = 0$ or $z_2 = 0$ (or both).

(5-5g) $\frac{z_1}{z_2} \cdot \frac{z_3}{z_4} = \frac{z_1 z_3}{z_2 z_4}$, if $z_2 \neq 0$, $z_4 \neq 0$.

(5-5h) $\frac{z_1}{z_2} + \frac{z_3}{z_4} = \frac{z_1 z_4 + z_2 z_3}{z_2 z_4}$, if $z_2 \neq 0$, $z_4 \neq 0$.

We leave the proofs of these relations as exercises. (See Exercises 5-5, Problems 7 - 9.)

Exercises 5-5

1. Find the multiplicative inverses of each of the following complex numbers and write them in standard form:
   - (a) 1
   - (b) 5
   - (c) i
   - (d) $-i$
   - (e) $1 + i$
   - (f) $2 + 3i$
   - (g) $1 + i^2$
   - (h) $4 - 3i$

2. Does every complex number have a multiplicative inverse?
3. What complex numbers are their own multiplicative inverses?
4. What complex numbers are the additive inverses of their multiplicative inverses?
5. Express the following quotients in standard form:

   (a) $\frac{1}{2 + i}$

   (b) $\frac{3}{2i}$

   (c) $\frac{1}{2i - 5}$

   (d) $\frac{13 + 5i}{2i}$

   (e) $\frac{1 + i}{2 - i}$

   (f) $\frac{4 + 3i}{2 + 5i}$

   (g) $\frac{7 + 6i}{3 - 4i}$

   (h) $\frac{3i}{4 + 7i}$

   (i) $\frac{-5i}{3 + 5i}$

   (j) $\frac{1 + \sqrt{2}i}{1 - \sqrt{2}i}$

   (k) $\frac{\sqrt{2} + \sqrt{3}i}{1 + \sqrt{2}i}$

   (l) $\frac{a + bi}{a - bi}$; a, b real, $a + bi \neq 0$

   (m) $\frac{a + 2bi}{2a - bi}$; a, b real, $2a - bi \neq 0$

   (n) $\frac{-m - ni}{-m + ni}$; m, n real, $-m + ni \neq 0$

   (o) $\frac{3x + 2yi}{x - yi}$; x, y real, $x - yi \neq 0$
6. Show that, if $z_1 \neq 0$, the equation $z_1 \cdot z = z_2$ has no more than one solution.
7. Write in standard form all complex numbers z such that the real part of $\frac{1}{z}$ is $\frac{1}{2}$, and
   (a) the imaginary part of z is zero.
   (b) the imaginary part of z is $\frac{1}{2}$.
   (c) the imaginary part of z is 1.
8. Prove that $z_1 z_2 = 0$ if and only if $z_1 = 0$, or $z_2 = 0$, or both are zero.

*9. Prove that $\frac{z_1}{z_2} \cdot \frac{z_3}{z_4} = \frac{z_1 z_3}{z_2 z_4}$, if $z_2 \neq 0$, $z_4 \neq 0$.

*10. Prove that $\frac{z_1}{z_2} + \frac{z_3}{z_4} = \frac{z_1 z_4 + z_2 z_3}{z_2 z_4}$, if $z_2 \neq 0$, $z_4 \neq 0$.

11. Make use of the formulas in Problems 9 and 10 to obtain the following sums and products. Write the answers you obtain in standard form.

   (a) $\frac{1 + i}{1 + 2i} + \frac{1 - i}{1 - 2i}$

   (b) $\frac{1 + 2i}{3 + 4i} \cdot \frac{2-i}{2i}$

   (c) $\frac{2 + 36i}{6 + 8i} + \frac{7 - 26i}{3 - 4i}$

   (d) $\frac{2 - 3i}{3 + 2i} + \frac{3 + 4i}{2 - 4i}$

   (e) $(\frac{a + bi}{a - bi})^2 + (\frac{a - bi}{a + bi})^2 \quad a + bi \neq 0, \ a - bi \neq 0$

*12. Show that the words "in standard form" may be omitted in Theorem 5-5 if we suppose merely that $a^2 + b^2 \neq 0$.

5-6. Quadratic Equations.

We come now to a crucial test for the complex number system. Does it permit us to solve equations of the form

$$az^2 + bz + c = 0, \tag{5-6a}$$

where a, b, c are real numbers and

$$b^2 - 4ac < 0 \ ? \tag{5-6b}$$

Let us first find the solutions of the quadratic equation on which we have so far focussed our attention:

$$z^2 + 1 = 0. \tag{5-6c}$$

If z is an arbitrary complex number, we have

$$z^2 + 1 = z^2 - (-1) = z^2 - i^2 = (z - i)(z + i).$$

This factorization of $z^2 + 1$ shows that if z is a complex number satisfying Equation (5-6c), then one of the factors $(z - i)$, $(z + i)$ must be zero, and z must be either i or $-i$. Converseely, we see that i and $-i$ both satisfy Equation (5-6c). Therefore we conclude that the solutions of Equation (5-6c) are $i, -i$.

Equation (5-6c) is a special case of the Equation

$$z^2 = r. \tag{5-6d}$$

From Chapter 1, we know that if r is real and positive this equation has two real solutions. We have just seen that for a special negative value of r, namely $r = -1$, this equation has two non-real complex solutions, i and $-i$. Let us next consider the general case in which r is negative.

If r is real and negative then $-r$ is real and positive, and $\sqrt{-r}$ is defined. We have

$$r = (-1)(-r) = (i)^2 (\sqrt{-r})^2 = (i\sqrt{-r})^2;$$

and hence

$$z^2 - r = z^2 - (i\sqrt{-r})^2 = (z - i\sqrt{-r})(z + i\sqrt{-r}).$$

Just as in the discussion of Equation (5-6c), we conclude that Equation (5-6d) has the two solutions $i\sqrt{-r}$, $-i\sqrt{-r}$, when r is real and negative.

For the case in which r is real and positive we introduced the notation $\sqrt{r}$ to describe the solution set of Equation (5-6d): one solution is $\sqrt{r}$ and the second $-\sqrt{r}$. It would be desirable to extend the definition of $\sqrt{r}$ for negative real r so that the description of the solution set of Equation (5-6d) would be the same for all r. The question is which of the two solutions $i\sqrt{-r}$, $-i\sqrt{-r}$ shall we take to be $\sqrt{r}$ if r is negative?

Recall that in Chapter 1 we faced the problem of defining $\sqrt{r}$ unambiguously for positive r. The problem was resolved by defining $\sqrt{r}$ to be the non-negative solution of Equation (5-6d). The requirement that $\sqrt{r}$ be non-negative was simply an agreement adopted to make the meaning of $\sqrt{r}$ definite. However, this agreement makes no sense if the solutions of Equation (5-6d) are complex. We have not defined "positive" and "negative" for non-real complex numbers, and cannot define these terms for complex numbers in a way which is consistent with their usual meaning. We must make a new agreement for the case of negative r. Any agreement which definitely selects one of the solutions $i\sqrt{-r}$, $-i\sqrt{-r}$ of Equation (5-6d) will be satisfactory. We choose $\sqrt{r} = i\sqrt{-r}$, and accordingly make the following definition:

Definition 5-6a. Let r be any real number. We define $\sqrt{r}$ as follows:

(1) If $r \geq 0$, then $\sqrt{r}$ is the unique non-negative real number w such that $w^2 = r$.

(2) If $r < 0$, then $\sqrt{r} = i\sqrt{-r}$.

Example 5-6a.

$$\sqrt{-1} = i\sqrt{1} = i$$

$$\sqrt{-12} = i\sqrt{12} = 2\sqrt{3}\,i$$

$$\sqrt{(-2i)^2} \quad \sqrt{4i^2} = \sqrt{-4} = i\sqrt{4} = 2i.$$

Example 5-6b. Find the product $(\sqrt{-5})(\sqrt{-15})$.

Solution: We have

$$(\sqrt{-5})(\sqrt{-15}) = (i\sqrt{5})(i\sqrt{15}) = i^2\sqrt{5}\sqrt{15} = -\sqrt{75}.$$

Note that it is not correct to say

$$(\sqrt{-5})(\sqrt{-15}) = \sqrt{(-5)(-15)} = \sqrt{75}.$$

The statement $\sqrt{r}\sqrt{s} = \sqrt{rs}$ has been proved only for the case in which r and s are both positive. The statement is also true if r and s have opposite signs (Exercises 5-6, Problem 5), but as the foregoing example shows, it is false if both r and s are negative.

Example 5-6c. Find the product $(\sqrt{r})(\sqrt{r^3})$ if r is any real number.

Solution: We have to consider two cases. If $r \geq 0$ we have $r^3 \geq 0$, and

$$\sqrt{r}\sqrt{r^3} = \sqrt{r \cdot r^3} = \sqrt{r^4} = r^2.$$

If $r < 0$ we have $r^3 < 0$, and

$$\sqrt{r}\sqrt{r^3} = (i\sqrt{-r})(i\sqrt{-r^3}) = i^2\sqrt{(-r)(-r^3)} = -\sqrt{r^4} = -r^2.$$

Now that we have given an unambiguous meaning to $\sqrt{r}$ for each real number r, we state as a theorem our previous conclusions about equations of the form $z^2 = r$, where r is any given real number.

Theorem 5-6a.

If r is any given real number, the equation $z^2 = r$ has the roots $\sqrt{r}$ and $-\sqrt{r}$, and no others.

We now turn to the solution of the general quadratic equation

(5-6e) $$az^2 + bz + c = 0, \quad a, b, c \text{ real and } a \neq 0.$$

Recall that we were led to our study of complex numbers because we failed to find real solutions of Equation (5-6e) when its discriminant $b^2 - 4ac$ is negative. However, reasoning just as in Chapter 4, we prove the following theorem:

Theorem 5-6b.

The equation

$$az^2 + bz + c = 0, \quad a, b, c \text{ real and } a \neq 0,$$

has the solutions

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a}, \qquad \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

and no others.

There is nothing new if $b^2 - 4ac \geq 0$; this is the case of real solutions discussed in Chapter 4. We now prove that the formula holds if $b^2 - 4ac < 0$, although in this case the solutions will not be real.

The proof is the same as in Chapter 4. Recall the procedure: divide by a and complete the square.

$$z^2 + \frac{b}{a} z + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2},$$

(5-6f) $$\left(z + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}.$$

We now have Theorem 5-6a which tells us that Equation (5-6f) has (complex) solutions whether $b^2 - 4ac$ is positive, negative, or zero.

Applying Theorem 5-6a, we obtain

$$z + \frac{b}{2a} = \sqrt{\frac{b^2 - 4ac}{4a^2}} \quad \text{or} \quad z + \frac{b}{2a} = -\sqrt{\frac{b^2 - 4ac}{4a^2}};$$

so $$z = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{or } z = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

The proof of Theorem 5-6b can be completed by showing that the numbers obtained actually satisfy the equation.

Example 5-6d. Find the solutions of $z^2 + z + 1 = 0$.

Solution: $a = b = c = 1$. By Theorem 5-6b the solutions are

$$\frac{-1 + \sqrt{-3}}{2} = \frac{-1 + i\sqrt{3}}{2},$$

and

$$\frac{-1 - \sqrt{-3}}{2} = \frac{-1 - i\sqrt{3}}{2}.$$

Other statements about the relation between the solutions and coefficients of a quadratic equation can be established just as in Chapter 4. In particular, if z_1 and z_2 are the complex solutions of the equation

$$az^2 + bz + c = 0,$$

then

(5-6g) $$z_1 + z_2 = \frac{-b}{a}, \quad z_1 \cdot z_2 = \frac{c}{a};$$

and

(5-6h) $$az^2 + bz + c = a(z - z_1)(z - z_2).$$

The proofs are left as exercises.

Exercises 5-6

1. Perform the indicated operations and write the answers you obtain in standard form.
   - (a) $\sqrt{-25} + \sqrt{-4}$
   - (b) $-\sqrt{-5} - 6\sqrt{-20}$
   - (c) $\sqrt{-2} + 5\sqrt{-8} - \sqrt{-98}$
   - (d) $\sqrt{-4} \cdot \sqrt{-5}$
   - (e) $\sqrt{-6} \cdot \sqrt{-8} \cdot \sqrt{-1}$
   - (f) $\sqrt{-\frac{1}{4}}$
   - (g) $\dfrac{\sqrt{-81}}{3\sqrt{-8}}$
   - (h) $\sqrt{\dfrac{-98}{-147}}$

2. Write each of the following complex numbers in standard form. Assume c is a real number.
   - (a) $\sqrt{-(2)^2}$
   - (b) $\sqrt{(-2)^2}$
   - (c) $\sqrt{-(-2)^2}$
   - (d) $\sqrt{c^2}$
   - (e) $\sqrt{(-c)^2}$
   - (f) $\sqrt{-c^2}$
   - (g) $\sqrt{-(-c)^2}$

3. Perform the indicated operations and write the answers you obtain in standard form. Assume a and b are positive real numbers.

(a) $\sqrt{-a^2} + \sqrt{-b^2}$ (d) $\dfrac{5\sqrt{-a^2}}{3\sqrt{-a}}$

(b) $\sqrt{-a^2} \cdot \sqrt{-4a^2b}$ (e) $\sqrt{-32a^2} - \sqrt{-50a^2}$

(c) $\sqrt{-a}\,(\sqrt{-a} + \sqrt{-b})$ (f) $\sqrt{-a} \cdot \sqrt{-a^3}$

(g) $\sqrt{-a^2 - 2ab - b^2} + \sqrt{-(a + b)^2}$

*4. Examine the proof that $\sqrt{ab} = \sqrt{a}\sqrt{b}$ if a and b are non-negative real numbers, and explain why the same argument cannot be used when a and b are negative.

5. Show that if $r < 0$ and $s > 0$, then $\sqrt{r}\sqrt{s} = \sqrt{rs}$.

In each of Problems 6 - 17 solve the given quadratic equation and express the solutions in standard form.

6. $z^2 + 1 = 0$ 12. $z^2 - 4z + 8 = 0$

7. $z^2 + z - 1 = 0$ 13. $2z^2 + z + 1 = 0$

8. $z^2 + 2z + 2 = 0$ 14. $z^2 - 4z - 8a = 0$ (a real)

9. $z^2 - z + 1 = 0$ 15. $mz^2 + z + \frac{1}{m} = 0$ (m real, $m \neq 0$)

10. $3z^2 + 2z + 4 = 0$ 16. $z^2 - iz + 2 = 0$

11. $z^2 + 4z + 8 = 0$ 17. $az^2 + c = 0$ (a, c real, $a \neq 0$)

18. The equation $z^3 - 8 = 0$ has the solution 2. Show that $z - 2$ is a factor of $z^3 - 8$, and use this fact to find two more solutions of the equation.

19. Suppose z_1 and z_2 are the solutions of $az^2 + bz + c = 0$, where a, b, c are real and $a \neq 0$. Show that

$z_1 + z_2 = -\frac{b}{a}$ and $z_1 z_2 = \frac{c}{a}$.

*20. If z_1 and z_2 are the solutions of the equation $az^2 + bz + c = 0$ show that the equation

$$az^2 + bz + c = a(z - z_1)(z - z_2)$$

holds for every element z of C. (This formula therefore provides a "factorization" of the expression $az^2 + bz + c$.)

21. Find quadratic equations which have the following pairs of solutions:

    (a) $z_1 = 1 - i$, $z_2 = 1 + i$

    (b) $z_1 = i$, $z_2 = 2 + i$

    (c) $z_1 = 0$, $z_2 = 0$

    (d) $z_1 = a_1 + b_1 i$, $z_2 = a_2 + b_2 i$; a_1, b_1, a_2, b_2 being any four given real numbers.

*22. Solve the equation $z^2 = i$. [Hint: Writing z in standard form, $z = x + yi$, the given equation is equivalent to a pair of equations whose unknowns are real: $x^2 - y^2 = 0$, $2xy = 1$.]

*23. Solve the equation $z^2 = -i$.

*24. Find an equation whose solutions are $1 + 2i$, $1 - i$, $1 + i$. Is there a quadratic equation having these numbers as solutions? If there is one, find it. If there is none, prove that there is none.

5-7. Graphical Representation; Absolute Value.

According to Property C-4 and Theorem 5-4a each complex number z may be written in one and only one way in the standard form $a + bi$, where a and b are real numbers. Thus each complex number z determines, and is determined by, an ordered pair (a,b) of real numbers: a is the real part of z, b the imaginary part of z.

Recalling that ordered pairs of real numbers formed the starting point of coordinate geometry, we find that we are able to represent the complex numbers by points in the xy-plane. Agreeing to associate z with the point (a,b) if and only if $z = a + bi$, in standard form, we set up a one-to-one correspondence between the elements of C and the points of the xy-plane.

It is customary to use the expression "Argand diagram" to describe the pictures obtained when the point (a,b) of the xy-plane is used to represent the complex number a + bi given in standard form. Figure 5-7a is an example of an Argand diagram showing three points (0,0), (4,-5), (-4,3) and the complex numbers they represent. Note that points on the x-axis correspond to real numbers and points on the y-axis correspond to pure imaginary numbers. For the sake of brevity we shall often say "the point z = x + yi" instead of "the point (x,y) corresponding to the complex number z = x + yi."

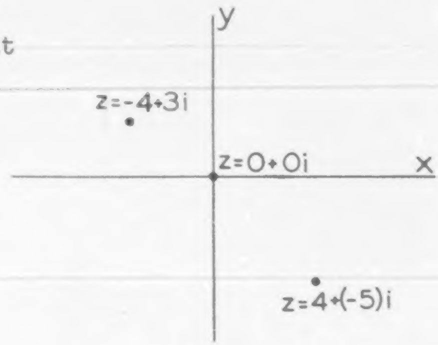


Figure 5-7a

The geometric representation of complex numbers by means of an Argand diagram serves a double purpose. It enables us to interpret statements about complex numbers geometrically and to express geometric statements in terms of complex numbers. As a first example, consider the formula established in Chapter 2 for the coordinates of the midpoint of a line segment: The midpoint of the segment joining (x_1,y_1) and (x_2,y_2) is the point (x,y) given by the formulas

$$x = \frac{x_1 + x_2}{2}, \quad y = \frac{y_1 + y_2}{2} . \tag{5-7a}$$

In terms of complex numbers this may be stated: The midpoint of the segment joining the points $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ is the point z = x + yi given by

$$z = \frac{z_1 + z_2}{2} . \tag{5-7b}$$

Note that we can express in one "complex" equation a statement which requires two "real" equations.

Now we can use Equation (5-7b) to establish a geometric interpretation of addition of complex numbers. Let z_1 and z_2 be two complex numbers and suppose that the points 0, z_1, z_2 are not collinear. Let $z_3 = z_1 + z_2$ and consider the quadrilateral whose vertices are 0, z_1, z_2, z_3 (Figure 5-7b). The midpoint of the diagonal from z_1 to z_2 is $\frac{z_1 + z_2}{2}$; that of the diagonal from 0 to z_3 is

$$\frac{0 + z_3}{2} = \frac{z_3}{2} = \frac{z_1 + z_2}{2} .$$

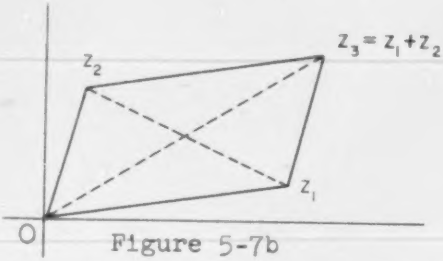


Figure 5-7b

Hence the diagonals have a common midpoint. Since the diagonals of the quadrilateral bisect each other the figure is a parallelogram. Thus we have a geometrical construction for the sum of two complex numbers: If two complex numbers are plotted in an Argand diagram their sum is the complex number corresponding to the fourth vertex of the parallelogram whose other three vertices are the origin and the two given points (and which has the segments joining z_1 and z_2 to the origin as sides.)

When the points 0, z_1, z_2 are collinear the parallelogram collapses into a straight line and our construction fails. We shall discuss this case later.

Next we consider the geometric construction of the difference $z_2 - z_1$ of two complex numbers. Since $z_2 - z_1 = z_2 + (-z_1)$ we have only to find a geometric construction of the additive inverse $-z$ of the complex number z. By equation (5-7b) the midpoint of the segment joining z and $-z$ is

$$\frac{z + (-z)}{2} = \frac{0}{2} = 0,$$

that is, the midpoint is the origin. Thus, if a complex number is plotted in an Argand diagram, its additive inverse is the complex number corresponding to the point symmetric to the given point with respect to the origin (Figure 5-7c).

We could now describe geometric constructions for the product and quotient of complex numbers but these constructions are not very illuminating. After we have studied trigonometry and the relation between complex numbers and trigonometry (Chapter 12) we will be able to state simple and elegant geometric interpretations of these operations.

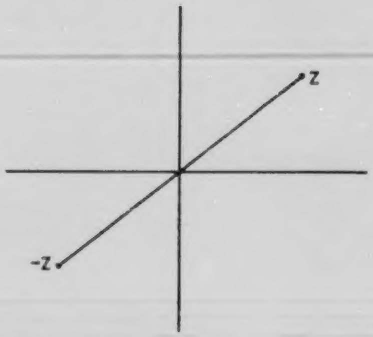


Figure 5-7c

Example 5-7a. Given $z_1 = 3 + i$, and $z_2 = 2 - 2i$, make use of an Argand diagram to find the difference $z_1 - z_2$.

Solution: Begin by plotting z_1 and z_2. Then locate the additive inverse of z_2, namely $-z_2$. This is easily done since we know that z_2 and $-z_2$ are symmetric with respect to the origin. The point $z_1 - z_2$ is the same as $z_1 + (-z_2)$. (See Figure 5-7d.)

The geometric representation of complex numbers suggests a definition of absolute value of a complex number. Recall that when real numbers are represented by points on a line the absolute value of a real number is equal to its distance from the origin. Accordingly we define the absolute value $|z|$ of a complex number $z = a + bi$ to be the distance from the origin of the point (a,b). Using the distance formula from Chapter 2 our definition may be stated algebraically as follows:

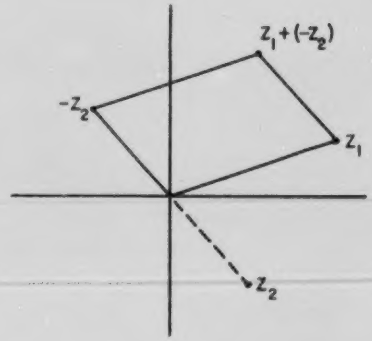


Figure 5-7d

Definition 5-7a. If $z = a + bi$, where a and b are real numbers we write

$$|z| = \sqrt{a^2 + b^2},$$

and call $|z|$ the absolute value of z.

Example 5-7b. Show that the distance between the points z_1 and z_2 is $|z_2 - z_1|$.

Solution: If $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ where x_1, y_1, x_2, y_2 are real numbers, then by the theorem on subtraction

$$z_2 - z_1 = (x_2 - x_1) + (y_2 - y_1)i.$$

By the definition of absolute value

$$|z_2 - z_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

and this is the distance between the points (x_1, y_1) and (x_2, y_2).

When z_1 and z_2 are real numbers we know the following relations involving absolute value and the algebraic operations:

$$|z_1 \cdot z_2| = |z_1|\ |z_2| \tag{5-7c}$$

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \tag{5-7d}$$

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{5-7e}$$

$$\Big|\, |z_1| - |z_2| \,\Big| \le |z_1 - z_2|. \tag{5-7f}$$

These relations continue to hold when z_1 and z_2 are complex numbers. Formulas (5-7c) and (5-7d) can be proved by calculation (Exercises 5-7, Problems 8 - 9), although we will present simpler proofs in the next section.

The algebraic proof of Formula (5-7c) is quite difficult but we can give an easy geometric proof. Consider the triangle whose vertices are 0, z_1, $z_1 + z_2$ in Figure 5-7b. The lengths of its sides are $|z_1|$, $|z_2|$, $|z_1 + z_2|$. Why? Since the length of a side of a triangle is less than the sum of the lengths of the other two sides, we have

$$|z_1 + z_2| < |z_1| + |z_2|.$$

We will show later that when the parallelogram collapses into a straight line we have either the inequality above or the equation

$$|z_1 + z_2| = |z_1| + |z_2| .$$

This will complete the proof of Formula (5-7e), which is often called the "triangle inequality". The discussion of (5-7f) is left as an exercise (Exercises 5-7, Problem 10).

For further discussion of the algebra and geometry of complex numbers it is convenient to introduce the notion of complex conjugate. We do this in the next section.

Exercises 5-7

1. Plot each of the following complex numbers in an Argand diagram. Label the points with the symbols z_1, z_2, etc.

   $z_1 = 1$ $\qquad$ $z_5 = 2 + i$

   $z_2 = i$ $\qquad$ $z_6 = -4 - 2i$

   $z_3 = -1$ $\qquad$ $z_7 = \sqrt{2} - i$

   $z_4 = -i$ $\qquad$ $z_8 = \pi - \sqrt{3}\, i$

2. Plot the additive inverse of each complex number in Problem 1. Label the point that corresponds to z_1 with the symbol $-z_1$, etc.

3. In each of the following problems find $z_1 + z_2$ and $z_1 - z_2$, and also construct them graphically.

   (a) $z_1 = 1 + i,$ $\qquad$ $z_2 = 2 + i$

   (b) $z_1 = 3 + 2i,$ $\qquad$ $z_2 = 2 + 3i$

   (c) $z_1 = -1 + 2i,$ $\qquad$ $z_2 = 2 - i$

   (d) $z_1 = -3 + 4i,$ $\qquad$ $z_2 = -1 - 3i$

   (e) $z_1 = -3 + i,$ $\qquad$ $z_2 = 1 + 4i$

   (f) $z_1 = -2i$ $\qquad$ $z_2 = 2 - 4i$

   (g) $z_1 = 3,$ $\qquad$ $z_2 = -3 + 5i$

   (h) $z_1 = 4,$ $\qquad$ $z_2 = -4i$

4. Let $z_1, z_2, \cdots, z_8$ be the points given in Problem 1. Use Equation 5-7b to find the midpoints of the segments joining

z_2 and z_5, z_3 and z_6, z_4 and z_7, and plot the points in an Argand diagram.

5. Find $|z|$ if:
   (a) $z = 3 - 4i$
   (b) $z = -2i$
   (c) $z = 1 + i^2$
   (d) $z = i^4 + i^7$
   (e) $z = \pi + \sqrt{2}\, i$
6. Show that, if $z \neq 0$, $\left| \frac{z}{|z|} \right| = 1$.
7. Find the set of points described by each of the following equations
   (a) $z = 1$ (b) $z = |z|$ (c) $z = \frac{z}{|z|}$
8. Give an algebraic proof of the equation
$$|z_1 z_2| = |z_1| \cdot |z_2|,$$
   if z_1 and z_2 are complex numbers.
9. Give an algebraic proof of the equation
$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|},$$
   if z_1 and z_2 are complex numbers, and $z_2 \neq 0$.
10. State a geometric proof of the inequality
$$\big| |z_1| - |z_2| \big| \leq \left| z_1 - z_2 \right|.$$
11. Suppose 0, $z_1 = a + bi$, $z_2 = c + di$ are collinear. If $z_3 = z_1 + z_2$ show that z_3 lies on the line through 0, z_1 and z_2.
12. Prove that the triangle with vertices 0, 1, z is similar to the triangle with vertices 0, z, z^2 by showing that corresponding sides are proportional. (Hint: Note that the length of each side of the second triangle is equal to $|z|$ multiplied by the length of each side of the first triangle.) Use the result to describe a geometric construction for z^2.

5-8. Complex Conjugate.

Definition 5-8a. If $z = a + bi$, in standard form (a and b real), we call $a + (-b)i$ the complex conjugate, or simply the conjugate of z, and write

$$\bar{z} = \overline{a + bi} = a + (-b)i.$$

Since $a + (-b)i = a - bi$ we may also write

$$\bar{z} = \overline{a + bi} = a - bi$$

Example 5-8a. $\overline{2 + 3i} = 2 - 3i$; $\overline{\left(\frac{1}{i}\right)} = -\bar{i} = i$.

It is easy to see that the conjugate of the conjugate of a complex number is the complex number itself. If $z = a + bi$ in standard form, we have

$$\overline{(\bar{z})} = \overline{(\overline{a + bi})} = \overline{a - bi} = a + bi$$

so that

(5-8a) $$\bar{\bar{z}} = z.$$

Thus if the first of two numbers is the conjugate of the second, then the second is the conjugate of the first. We call such a pair of numbers conjugate.

Although we have not used the term "conjugate" before, conjugates of complex numbers have appeared in many of our statements about complex numbers. Thus, for example, the solutions of a quadratic equation with negative discriminant are conjugate. Also, the formula for the multiplicative inverse of $z = a + bi$ can be written

$$\frac{1}{a + bi} = \frac{a + (-b)i}{a^2 + b^2} = \frac{\bar{z}}{|z|^2}$$

or

(5-8b) $$\frac{1}{z} = \frac{\bar{z}}{|z|^2}.$$

From Equation (5-8b) we get immediately

(5-8c) $$z \cdot \bar{z} = |z|^2.$$

This last equation is important enough to deserve statement as a theorem and a new proof.

Theorem 5-8a.

$$z \cdot \bar{z} = |z|^2.$$

Proof: If $z = a + bi$ in standard form, then

$$z \cdot \bar{z} = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - b^2 i^2 = a^2 - b^2(-1)$$
$$= a^2 + b^2 = (\sqrt{a^2 + b^2})^2 = |z|^2.$$

Now that we have proved Equation (5-8c) independently of Equation (5-8b) we can derive (5-8b) from (5-8c). In fact, it is convenient to use Theorem 5-8a directly in dividing complex numbers. The following example is illustrative.

Example 5-8b. Find the quotient $\frac{8 + 5i}{2 + 3i}$.

Solution: The conjugate of $2 + 3i$ is $2 - 3i$. Multiplying $\frac{8 + 5i}{2 + 3i}$ by $\frac{2 - 3i}{2 - 3i}$, and using Theorem 5-8a and Equation 5-5g, we get

$$\frac{8 + 5i}{2 + 3i} = \frac{2 - 3i}{2 - 3i} \cdot \frac{8 + 5i}{2 + 3i} = \frac{(2 - 3i)(8 + 5i)}{(2 - 3i)(2 + 3i)}$$
$$= \frac{(2)(8) - (5)(-3) + [8(-3) + 2(5)]i}{2^2 + 3^2}$$
$$= \frac{31}{13} + \left(\frac{-14}{13}\right)i = \frac{31}{13} - \frac{14}{13}\,i.$$

If we plot z and $\bar{z}$ in an Argand diagram (Figure 5-8a),

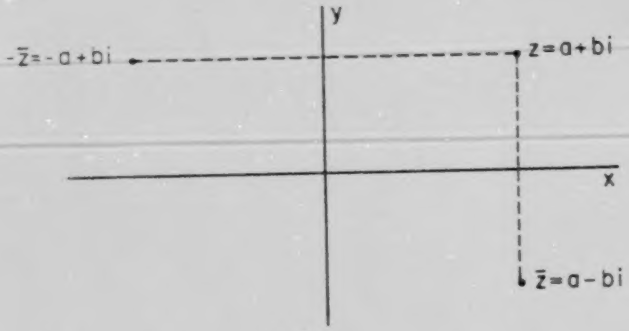


Figure 5-8a

we see that $\bar{z}$ is the reflection of z in the x-axis; that is, z and $\bar{z}$ are symmetric with respect to the x-axis. Similarly, $-\bar{z}$ is the reflection of z in the y-axis. From this diagram, or by direct calculation, we also see that $z + \bar{z} = 2a$ and $z - \bar{z} = 2bi$. With these equations we can express a and b in terms of z and $\bar{z}$. We thus obtain the following theorem:

Theorem 5-8b.

If $z = a + bi$ in standard form, then

$$z + \bar{z} = 2a, \quad z - \bar{z} = 2bi;$$

or

$$a = \frac{1}{2}(\bar{z} + z), \quad b = \frac{1}{2}(\bar{z} - z).$$

Observe that since a complex number is real if and only if its imaginary part is 0 and pure imaginary if and only if its real part is 0, Theorem 5-8b has the following corollary.

Corollary. The complex number z is real if and only if $z = \bar{z}$ and pure imaginary if and only if $z = -\bar{z}$.

Theorem 5-8b permits us to state any relation between the real and imaginary parts of a complex number z as a relation between z and $\bar{z}$. In particular, every statement of analytic geometry can be expressed as a relation of this kind. Before considering examples we state the following theorem which simplifies the computation of conjugates.

Theorem 5-8c.

If z_1 and z_2 are any complex numbers, then

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$;

(b) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$;

(c) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$;

(d) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$.

The proofs are left as exercises (Exercises 5-8, Problem 5).

Example 5-8c. Show that, for any z, the reflection of the point $3iz + 2$ in the x-axis is the point $-3i\bar{z} + 2$.

Solution: The reflection of a point $3iz + 2$ in the x-axis is its conjugate, $\overline{3iz + 2}$. Using Theorem 5-8c twice we obtain

$$\overline{3iz + 2} = \overline{(3i)(z)} + \overline{2} = \overline{(3i)}(\overline{z}) + \overline{2}$$
$$= -3i\overline{z} + 2,$$

which was to be shown.

Example 5-8d. Show that the circle of radius 1 with center at the origin is the set of all points z which satisfy the equation

$$z \cdot \overline{z} = 1.$$

Solution: There are two possible approaches. We can start with the definition of this circle as the set of points whose distance from the origin is 1, and use the fact that the distance of the point $z = x + yi$ from the origin is $|z|$. Then z is on the circle if and only if

$$|z| = 1.$$

Squaring both sides of this equation and using Theorem 5-8a we get

$$z \cdot \overline{z} = |z|^2 = 1.$$

However we can also start with the equation of the circle from analytic geometry:

$$x^2 + y^2 = 1.$$

If $z = x + yi$ then by Theorem 5-8b

$$x = \frac{1}{2}(z + \overline{z}), \quad y = \frac{i}{2}(z - \overline{z}).$$

Substituting for x and y, we obtain

$$\left[\frac{1}{2}(\overline{z} + z)\right]^2 + \left[\frac{i}{2}(\overline{z} - z)\right]^2 = 1,$$

or

$$\frac{1}{4}(\overline{z} + z)^2 - \frac{1}{4}(\overline{z} - z)^2 = 1.$$

Simplifying, we have

$$z \cdot \overline{z} = 1.$$

Example 5-8e. Show that the segments which join the points $z_1 = x_1 + y_1 i$ and $z_2 = x_2 + y_2 i$ to the origin are perpendicular if and only if the product $z_1 \cdot \overline{z}_2$ is pure imaginary.

Solution: Again, there are two approaches. We can either express the geometric conditions immediately in terms of z_1 and z_2, or state them first in terms of (x_1, y_1) and (x_2, y_2), and then use Theorem 5-8b. We will follow the first approach.

The segments joining z_1 and z_2 to the origin will be perpendicular if and only if the triangle with vertices 0, z_1, z_2 is a right triangle. By the Pythagorean Theorem this will be true if and only if

$$|z_1|^2 + |z_2|^2 = |z_1 - z_2|^2.$$

Using Theorems 5-8a and 5-8c this equation may be written

$$z_1\overline{z}_1 + z_2\overline{z}_2 = (z_1 - z_2)(\overline{z_1 - z_2}) = (z_1 - z_2)(\overline{z_1} - \overline{z_2})$$

$$z_1\overline{z_1} + z_2\overline{z_2} = z_1\overline{z_1} - z_1\overline{z_2} - z_2\overline{z_1} + z_2\overline{z_2} ;$$

$$0 = -z_1\overline{z}_2 - z_2\overline{z}_1$$

or, using Theorem 5-8c again and referring to Equation (5-8a),

$$z_1\overline{z}_2 = - \overline{z}_1 z_2 = - \overline{(z_1\overline{z}_2)}.$$

By the Corollary to Theorem 5-8b this equation can hold if and only if the product $z_1\overline{z_2}$ is pure imaginary.

Finally, we can use Theorems 5-8a and 5-8c to establish Formulas 5-7c, 5-7d. We do the first as an example.

Example 5-8f. Show that $|z_1 \cdot z_2| = |z_1|\ |z_2|$

Solution: Since the numbers in the equation which is to be established are positive it will suffice to prove $|z_1 \cdot z_2|^2 = |z_1|^2|z_2|^2$. (Why?) We have

$$\begin{aligned} |z_1 \cdot z_2|^2 &= (z_1 \cdot z_2)\ \overline{(z_1 \cdot z_2)} = (z_1 \cdot z_2)(\overline{z_1} \cdot \overline{z_2}) \\ &= (z_1 \cdot \overline{z_1})\ (z_2 \cdot \overline{z_2}) = |z_1|^2\ |z_2|^2. \end{aligned}$$

This completes the proof.

Exercises 5-8

1. Express the conjugate of each of the following complex numbers in standard form:

(a) $2 + 3i$	(d) -5	(g) πi^7
(b) $-3 + 2i$	(e) $-2i$	(h) $4 + i^6$
(c) $1 - i$	(f) $1 - i^5$	(i) $-3i + \sqrt{3}\, i^2$

2. Use conjugates to compute the following quotients. Write the answer in standard form.

(a) $\frac{2 + i}{1 + i}$	(g) $\frac{-5 + 6i}{-3 - 4i}$
(b) $\frac{i}{i + 3}$	(h) $\frac{3 - 6i}{2i}$
(c) $\frac{-1 + i}{2 + 3i}$	(i) $\frac{5}{3 + \sqrt{5}\, i}$
(d) $\frac{-4 + 3i}{2 + 5i}$	(j) $\frac{3 - \sqrt{-2}}{5 - \sqrt{-3}}$
(e) $\frac{7 + 6i}{3 - 4i}$	(k) $\frac{\sqrt{3} - \sqrt{-7}}{\sqrt{3} - \sqrt{-5}}$
(f) $\frac{3 + 2i}{4i}$	(l) $\frac{i^3 - 1}{i^2 - 1}$

(m) $\frac{a + bi}{2a + 3bi}$; a, b real, $2a + 3bi \neq 0$

(n) $\frac{x + yi}{2x - yi}$; x, y real, $2x - yi \neq 0$

(o) $\frac{(1 + i)(-1 + 2i) + (2 - i)}{2 - 3i}$

(p) $\frac{2i}{(i - 1)(i - 2)(i - 3)}$

3. For each of the following sketch in an Argand diagram the set of complex numbers z which satisfy the given equation.

(a) $z = \frac{1}{z}$ (b) $\overline{z} = \frac{1}{z}$

4. For each of the following sketch in an Argand diagram the set of points z that satisfies the given equation.

(a) $z + \overline{z} = 3$ (b) $z - \overline{z} = 2i$ (c) $z - \overline{z} = 3 + 2i$

5. Let $z_1 = x_1 + y_1 i$, $z_2 = x_2 + y_2 i$ be any complex numbers, x_1, y_1, x_2, y_2 real. Prove each of the following.

(a) $\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}$

(b) $\overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$

5. (c) $\overline{z_1 - z_2} = \overline{z_1} - \overline{z_2}$ [Hint: Show that $\overline{(-z_2)} = -(\overline{z_2})$ and use (a).]

(d) $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\overline{z_1}}{\overline{z_2}}$ [Hint: Show that $\overline{\left(\frac{1}{z_2}\right)} = \frac{1}{\overline{z_2}}$ and use (b).]

6. For any z, find the reflection of the point $z^3 - (3 + 2i)z^2 + 5iz - 7$ in the y-axis.

7. If $z^2 = (\overline{z})^2$, show that z is either real or pure imaginary.

8. Show that the product $z_1\overline{z_2}$ is pure imaginary if and only if $\frac{z_1}{z_2}$ is pure imaginary.

9. Prove that $|z_1 - z_2|^2 + |z_1 + z_2|^2 = 2|z_1|^2 + 2|z_2|^2$.

10. Suppose z_1 and z_2 are complex numbers and that

$$z_1 + z_2 \text{ and } z_1z_2$$

are real numbers. Show that either

$$z_1 \text{ and } z_2 \text{ are real,}$$

or

$$z_1 = \overline{z_2} .$$

11. Use the relation $z \cdot \overline{z} = |z|^2$ to show that

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|} .$$

12. Write the equation of the straight line $y = 3x + 2$ as an equation in z and $\overline{z}$.

13. Show that if $K \neq 0$ is any complex number and C is any real number then $K\overline{z} + \overline{K}z = C$ is the equation of a straight line.

14. Show that the points z_1 and z_2 are symmetric with respect to the line $y = x$ if and only if

$$(1 - i)\overline{z_1} + (1 + i)z_2 = 0.$$

15. What is the relation between the line segments joining z_1 and z_2 to the origin if the product $z_1\overline{z_2}$ is real?

5-9. Polynomial Equations

Linear and quadratic equations are special cases of polynomial equations. A polynomial is an expression of the form

(5-9a) $\quad P(z) = a_o z^n + a_1 z^{n-1} + \cdots + a_{n-2} z^2 + a_{n-1} z + a_n$

where n is a non-negative integer and $a_o, a_1, a_2, \cdots, a_{n-1}, a_n$ are any given complex numbers, $a_o \neq 0$. The non-negative integer n is called the degree of the polynomial and the numbers $a_o, a_1, a_2, \cdots, a_{n-1}, a_n$ are called its coefficients. A polynomial equation of degree n is an equation

(5-9b) $\quad P(z) = 0,$

where $P(z)$ is a polynomial of degree n. Linear equations are polynomial equations of degree 1; quadratic equations are polynomial equations of degree 2.

Examples 5-9a.

(a) $2z^3 - \frac{3}{5} z^2 + z - 2 = 0$ is a polynomial equation of degree 3 with rational coefficients.

(b) $z^5 - \sqrt{2} z^3 + 7z^2 - 3 = 0$ is a polynomial equation of degree 5 with real coefficients.

(c) $z^3 - 7\sqrt{z} + 3 = 0$ is not a polynomial equation.

(d) $5z^3 - (2 - i)z + (3 + 7i) = 0$ is a polynomial equation of degree 3 with complex coefficients.

(e) $z - 3 + \frac{1}{z^2} = 0$ is not a polynomial equation, but multiplying by z^2 we obtain the polynomial equation $z^3 - 3z^2 + 1 = 0$. Every solution of the first equation is a solution of the second, and every solution of the second equation is a solution of the first.

Every equation which can be written in terms of the unknown and given numbers, using only the operations of addition, multiplication, subtraction and division, can be transformed into a polynomial equation, equivalent except for extraneous roots. Thus, ordinary algebra is mostly concerned with the solution of polynomial equations. Let us summarize some of the advantages that the complex number system C has over the real number system R in connection with polynomial equations.

There are certain quadratic equations whose coefficients are in R but which have no solutions in R; every such equation has solutions in C. This was proved in Section 5-6 for the case of real coefficients, but it is true even if the coefficients are complex numbers. For example, the equation

$$z^2 + (1 - 5i)z - (12 + 5i) = 0$$

has the two solutions $2 + 3i$ and $-3 + 2i$, a fact which may be checked by substitution. Methods for finding such solutions will be presented in Chapter 12. The theorem that the solutions of any quadratic equation with complex coefficients are complex numbers is an unexpected and remarkable result. It shows us that we will not have to extend the complex number system in order to solve quadratic equations whose coefficients are in C. Recall that R does not have this property; indeed it was just for this reason that we extended R to C.

But the merits of C go far beyond this. Every polynomial equation with coefficients in C has solutions in C, and indeed all the solutions that could be expected are in C. This result, which is known as the Fundamental Theorem of Algebra, comes as an enormous bonus, when we recall that to solve the simple equation $x^2 = -1$ the new element i had to be invented. Conceivably, one might expect to need a new number j to solve $x^4 = -1$, for example. This is not the case! This equation has four and only four complex solutions, all of the form $a + bi$, where a and b are real numbers. (See Chapter 12 and Exercises 5-9.)

The first proof of the Fundamental Theorem was given by Gauss in 1799. Since then several other proofs have been developed and although some are quite simple, none is simple enough to be presented here. We shall however make a precise statement of the theorem in a form which is basic for the study of polynomials.

Theorem 5-9.

Let

$$P(z) = a_o z^n + a_1 z^{n-1} + \cdots + a_{n-2} z^2 + a_{n-1} z + a_n$$

be a polynomial of degree n with complex coefficients. Then there exist n complex numbers $r_1, r_2, \cdots, r_n$ (not necessarily distinct) such that

$$P(z) = a_o(z - r_1)(z - r_2)\cdots (z - r_n).$$

If one of the factors in the factorization of $P(z)$ stated in Theorem 5-9 is $z - r$, r is called a zero of $P(z)$; if exactly m of these factors are $z - r$, r is called a zero of multiplicity m. A zero is called a simple zero if its multiplicity is one; otherwise it is called a multiple zero. Since the total number of factors in Theorem 5-9 is n, the sum of the multiplicities of the zeros of a polynomial of degree n is n. This may also be stated: The number of zeros, each counted with its multiplicity, of a polynomial of degree n is n.

Since a product is 0 if and only if one of its factors is 0, it is clear that z is a solution of the polynomial equation

$$P(z) = 0$$

if and only if z equals one of the zeros of $P(z)$. According to Theorem 5-9 a polynomial of degree $n > 0$ has at least one zero (exactly one if $r_1 = r_2 = \cdots = r_n$) and may have as many as n zeros (exactly n if no two of the numbers $r_1, r_2, \cdots, r_n$ are equal). It follows that every polynomial equation of degree $n > 0$ has at least one complex solution, and may have as many as n solutions, but has no more than n solutions.

Example 5-9b. Discuss the possible number of solutions of a polynomial equation of degree 3. Include examples.

Solution: The equation may have 1, 2, or 3 solutions. If it has one solution, this must be a triple zero (zero of multiplicity 3) of the polynomial. If it has two solutions, one must be a simple zero, the other a double zero (zero of multiplicity 2) of the polynomial. If it has three solutions each must be a simple zero of the polynomial.

An example of the first case is given by the polynomial equation

$$z^3 - 3z^2 + 3z - 1 = (z - 1)^3 = 0.$$

The only solution of the equation is $z = 1$. 1 is a triple zero of the polynomial $z^3 - 3z^2 + 3z - 1$.

The equation

$$z^3 - z^2 - z + 1 = (z + 1)(z - 1)^2 = 0$$

has the solutions 1, -1. -1 is a simple zero and +1 a double zero.

The equation

$$z^3 + z = z(z - i)(z + i) = 0$$

has the solutions 0, i, $-i$. Each is a simple zero of $z^3 + z$.

Let $P(z)$ be a polynomial of degree n,

$$P(z) = a_o(z - r_1)(z - r_2)\cdots (z - r_n),$$

and define $Q(z)$ by

$$Q(z) = a_o(z - r_2)\cdots (z - r_n).$$

Then $Q(z)$ is a polynomial of degree $n - 1$ whose zeros are the zeros of $P(z)$, except possibly for r_1, and

$$P(z) = (z - r_1)Q(z).$$

Now suppose we have to determine the zeros of $P(z)$ and that we have found one zero, r_1. The remaining zeros will be the zeros of $Q(z)$ and to find $Q(z)$ we have only to divide $P(z)$ by $z - r_1$, since

$$\frac{P(z)}{z - r_1} = Q(z)$$

This fact enables us to reduce the solution of a polynomial equation of degree n to the solution of an equation of degree $n - 1$ once we have determined one solution of the original equation. The following example illustrates this.

Example 5-9c. Find all solutions of the equation $z^3 - 1 = 0$.

Solution: The solutions of the equation are the zeros of $z^3 - 1$. One zero is obviously 1. We divide $z^3 - 1$ by $z - 1$:

$$\begin{array}{r|l} & \underline{z^2 + z + 1\qquad} \\ z - 1 & z^3 \qquad\qquad -1 \\ & \underline{z^3 - z^2} \\ & \qquad z^2 \\ & \qquad \underline{z^2 - z} \\ & \qquad\qquad z \\ & \qquad\qquad \underline{z - 1} \\ & \qquad\qquad\qquad 0 \end{array}$$

The remaining solutions thus are the zeros of $z^2 + z + 1$, that is, the solutions of

$$z^2 + z + 1 = 0.$$

Solving this quadratic equation we get the roots $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$, $-\frac{1}{2} - i\frac{\sqrt{3}}{2}$. Thus the solutions of the given equation are 1, $\frac{1}{2} + \frac{i\sqrt{3}}{2}$, $\frac{1}{2} - \frac{i\sqrt{3}}{2}$.

In this example we observe that, as in the case of quadratic equations, the complex roots are conjugate. We can show that whenever the coefficients of a polynomial equation are real the complex solutions occur in conjugate pairs; that is, if z is a solution of such an equation $\overline{z}$ is also a solution. Let z be a solution of

$$a_oz^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n = 0.$$

Then we have $\overline{a_oz^n + a_1z^{n-1} + \cdots + a_{n-1}z + a_n} = \overline{0} = 0,$

and using Theorem 5-8c repeatedly we get

$$\overline{a}_o(\overline{z})^n + \overline{a}_1(\overline{z})^{n-1} + \cdots + \overline{a}_{n-1}(\overline{z}) + \overline{a}_n = 0.$$

Since the coefficients are real, $\overline{a}_o = a_o$, $\overline{a}_1 = a_1, \cdots, \overline{a}_{n-1} = a_{n-1}$, $\overline{a}_n = a_n$ and we have

$$a_o(\overline{z})^n + a_1(\overline{z})^{n-1} + \cdots + a_{n-1}\overline{z} + a_n = 0,$$

so that $\overline{z}$ is also a solution of the equation.

Exercises 5-9

1. Determine the zeros and the multiplicity of each zero for the following polynomials.
   (a) $5(z - 1)(z + 2)^3$
   (b) $z^4(z + \frac{1}{2})^2 (z - 3)$
   (c) $(z - 3 + 2i)^2(z + 1)^5$
2. Find the zeros of the following polynomials and state the multiplicity of each zero.
   (a) $z^5 + z^4 + 3z^3$
   (b) $z^4 + 2z^2 + 1$
   (c) $z^3 + 3z^2 + 3z + 1$
3. Write two polynomial equations whose only solutions are 1 and 2 such that:
   (a) the two equations have the same degree;
   (b) the two equations are of different degrees.
4. Discuss, with examples, the possible number of solutions of an equation of degree 4.
5. Find all solutions of $z^3 + 1 = 0$.
6. Find all solutions of the following equations, given one solution.
   (a) $3z^3 - 20z^2 + 36z - 16 = 0 \qquad z = 4$
   (b) $z^3 - 4z^2 + 6z - 4 = 0 \qquad z = 2$
7. Find all solutions of the following equations, given two solutions.
   (a) $z^4 + 2z^3 + z + 2 = 0 \qquad z = -1, \ -2$
   (b) $z^4 - 3z^3 - 3z^2 - 7z + 12 = 0 \qquad z = 4, 1$
8. Find the polynomial whose zeros include 1 and $-2i$ if:
   (a) the polynomial has the lowest possible degree.
   (b) the polynomial has real coefficients and has the lowest possible degree.
   (c) the polynomial has real coefficients, the lowest possible degree and $-2i$ is a double zero.

9. Given that $3 + \sqrt{2}\,i$ is a solution, find all solutions of the equation
$$z^4 - 6z^3 + 2z^2 + 54z - 99 = 0.$$

10. Given that $1 - \sqrt{5}i$ is a solution, find all solutions of the equation
$$z^4 - 2z^3 + 4z^2 + 4z - 12 = 0.$$

11. (a) Find a formula for the coefficients of the cubic polynomial whose zeros are r_1, r_2, r_3 if the coefficient of the highest power is 1.
   - *(b) Do the same for the quartic polynomial.
   - *(c) Make a guess as to the form of a corresponding formula for a polynomial of degree 7.

5-10. Miscellaneous Exercises.

1. If $z = 2 - 3i$, evaluate
$$-z,\ \bar{z},\ |z|,\ |\bar{z}|,\ \frac{1}{z},\ |z|^2,\ |z^2|, \text{ and } \frac{4 + 5i}{z}.$$

2. Write a quadratic equation having the solutions $c + di$ and $c - di$, where c and d are real.

3. Is the set of numbers $(1,-1,i,-i)$ closed with respect to multiplication? Addition?

4. If $z = x + yi$ show that
$$x \leq |z| \text{ and } y \leq |z|.$$

5. Sketch the set of points z which satisfy each of the following conditions.

(a) $|z - 2| = 3$ (c) $|z - 2i| < 4$

(b) $|z + 2| > 3$ (d) $|z - z_0| \leq 5$

6. Write an equation in x and y which is equivalent to the equation $|z - (2 + 3i)| = 5$.
Describe the set of points in an Argand diagram which satisfy the given equation.

7. Give a geometrical interpretation for the following relations.
   (a) $|z_1| < |z_2|$ (d) $z_1 + \overline{z}_2 = 0$
   (b) $|z| = 5$ (e) $z_1 - \overline{z}_2 = 0$
   (c) $z_1 + z_2 = 0$

8. Find all complex numbers z such that (Real part of z) = (Imaginary part of z), and $|z| = 1$.

9. Determine all quadratic equations with real coefficients which have $3 + 2i$ as a solution.

10. Plot the point corresponding to $3 + 5i$ in an Argand diagram. Then multiply the given number successively by i, i^2, and i^3, and plot the three points which correspond to the resulting products. Finally, show that the three last named points together with the given point form the vertices of a square.

11. Show that if z_o is a solution of the equation $az^2 + bz + c = 0$, where a, b, c are real and $b^2 - 4ac < 0$, then $z_o\overline{z}_o = \frac{c}{a}$ and $z_o + \overline{z}_o = -\frac{b}{a}$. Use the result to describe a geometric construction for z_o.

12. Find all quadratic equations with real coefficients having solutions z_1 and z_2 such that $z_1 + z_2 = 1$ and $z_1 z_2 = 4$.

13. Find all complex numbers z for which the real part of z^2 is 0. Show that if z belongs to this set, then $\frac{1}{z}$ also belongs to the set.

14. For what real values of r does the equation
$$rx^2 + (1 + r)x + 2 = 0$$
have non-real complex solutions? For what values of r does it have only one solution?

15. Show by an example that $a - bi$ need not be the complex conjugate of $a + bi$?

16. Find the equation of the perpendicular bisector of the line joining z_1 and z_2. (Hint: Use the fact that the perpendicular bisector of a line segment is the set of points equidistant from the endpoints.)

17. Let $z_o = x_o + y_o i$. Describe the set of points $z = x + yi$ which satisfy the inequality $\frac{|z - \bar{z}_o|}{|z - z_o|} < 1$.

18. Let z_1 and z_2 be distinct non-zero complex numbers. Show that z_1 and z_2 represent points in an Argand diagram lying on a straight line through the origin if and only if $\frac{z_1}{z_2}$ is real.

19. Solve the equation $z^4 = -1$. (You may find it helpful to refer to Exercises 5-6, Problems 22 and 23.)

20. Show that it is impossible to satisfy all the order postulates of Chapter 1 in the complex number system. Consider the element i. Certainly $i \neq 0$, so either $i > 0$ or $i < 0$ if the "Trichotomy" property is to hold. Show that each of the assumptions $i > 0$, $i < 0$ leads to conclusions contradicting at least one of the order postulates.

21. Find all complex numbers x,y with the property that the conjugate of $x + yi$ is $x - yi$.

*22. If $z = x + yi$, show that

$$|x| + |y| \leq \sqrt{2}\,|z|.$$

*5-11. Construction of the Complex Number System.

In this chapter we have assumed that we have available a number system (which we called the complex number system) satisfying certain imposed requirements (the four fundamental properties C-1, C-2, C-3, C-4). In a sense we have stated what a complex number system ought to be. On the basis of the imposed requirements we have learned how to compute in such a system.

It is a fundamental (but sophisticated) question whether there actually exists a number system C fulfilling the requirements we set down in Sections 5-1 and 5-2. We shall sketch the basic steps for constructing such a system. Many of the details will be left to the reader.

Let us return to our earlier developments. There we learned that the rule which associates with the complex number a + bi the ordered pair of real numbers (a,b) sets up a one-to-one correspondence between the set of complex numbers and the set of ordered pairs of real numbers. This fact and the information which we have obtained on how we are compelled to add and multiply in C motivates the following proposal for constructing, on the basis of the real number system, a number system which meets the requirements we imposed on C.

Let K denote the set of ordered pairs of real numbers (a,b). These are the objects which we are to "add" and "multiply". Let us say: (a,b) = (c,d) if and only if and only if a = c and b = d.

It is necessary to define operations of addition and multiplication for K. The facts we have deduced from the fundamental properties of the complex number system lead us to believe that the definitions which we shall put down are "reasonable" when we keep in mind our mission of constructing a complex number system with "real building blocks".

We define

Addition: $(a,b) + (c,d) = (a + c, b + d)$.

Multiplication: $(a,b)\cdot(c,d) = (ac - bd, ad + bc)$.

Note that the operation of "addition" in K is defined in terms of the operation of addition in the real number system and that the operation of "multiplication" in K is *defined* in terms of addition, subtraction and multiplication in the real number system. Note that our definitions assure closure of the operations $+$ and $\bullet$ of K: the "sum" of two ordered pairs of real numbers is an ordered pair of real numbers, the "product" of two ordered pairs of real numbers is an ordered pair of real numbers.

Two remarks are in order. First, we must distinguish, "addition" and "multiplication" in K from addition and multiplication in the real number system. The two kinds of addition and multiplication apply respectively to different kinds of objects. That is why we use the exaggerated plus sign $+$ and the exaggerated times sign $\bullet$ for the operations of "addition" and multiplication" in K.

Second, we emphasize that $+$ and $\bullet$ are constructed from what we learned about addition and multiplication in C keeping in mind that our correspondence between $a + bi$ and (a,b) identifies "real part" with "first component" and "imaginary part" with "second component". The spadework sets in at this stage. We verify first that K with the addition $+$ and multiplication $\bullet$ satisfies the usual laws of algebra. This verification depends upon properties satisfied by the real number system. We easily verify that $(0,0)$ is the *additive identity* for K, that $(1,0)$ is the *multiplicative identity* for K, and that $(-1,0)$ is the additive inverse of the multiplicative identity.

Explicitly, we have the following results:

$$(a,b) + (0,0) = (a,b), \qquad (a,b) \bullet (1,0) = (a,b),$$

$$(1,0) + (-1,0) = (0,0).$$

Verify these three statements.

Further
$$(0,1) \bullet (0,1) = (-1,0).$$
Hence K possesses an element whose square is the additive inverse of the multiplicative identity. This sounds a bit heavy-handed but tells us that we have grounds for optimism as far as capturing something that will play the role of the all-important i.

Let us go so far as to denote (0,1) by i. We may write

$$(5\text{-}11a) \quad (a,b) = (a,0) + (0,b) = (a,0) + (b,0) \cdot (0,1) = (a,0) + (b,0) \cdot i$$

Now if we restrict our attention to the special elements of K whose second components are zero, we see that they behave under + and • the same way that their first components do under the + and • of the real number system. That is

$$(5\text{-}11b) \qquad (a,0) + (b,0) = (a + b,0),$$

$$(5\text{-}11c) \qquad (a,0) \cdot (b,0) = (ab,0).$$

Verify the statements (5-11b), (5-11c) and also the following two:

$$(a,0) + (-a,0) = (0,0);$$

$$(a,0) \cdot (\tfrac{1}{a}, 0) = (1,0), \qquad a \neq 0.$$

We now define a notion of order among the special elements of the form (a,0). (Remark: We could not define a notion of order in K, even if we wanted to, which would yield the expected relation among the special elements (a,0). This remark applies to C also. If we had an order relation in C like that in R we could expect the square of each non-zero element to be positive. This would force i^2 into the unacceptable position of being both positive and negative in the sense of the real number system.) We define

"Less than": $[(a,0) < (b,0)]$ means $(a < b)$.

It is now possible to show that the set of elements of the form (a,0) together with the operation of addition + , the operation of multiplication • , and the relation of inequality < satisfy the postulates for the real number system.

Verify this assertion.

We are thus justified in taking this set of awkward appearing elements (a,0) with addition, multiplication and order so introduced as our real number system. With this understanding we verify that K has all the properties imposed on C. Note that (-1,0) is the additive inverse of the multiplicative identity of our present real number system and that

$$i \cdot i = (-1,0). \tag{5-11d}$$

Thanks to the fact that the elements $(a,0)$ may be taken as the real numbers, Property C-2 is satisfied. By Formula (5-11d), Property C-3 is satisfied. Further Formula (5-11a) tells us that Property C-4 is satisfied. There remains to be verified only that $+$ and $\cdot$ are commutative and associative, that the distributive law holds in K, and that each element has an additive inverse, in order to show that K has Property C-1.

The commutative and associative laws for $+$ and $\cdot$ are readily verified as is the distributive law. As an illustration we consider the distributive law:

$$\begin{aligned} &(a,b) \cdot [(c,d) + (e,f)] \\ &= (a,b) \cdot (c + e, d + f) \\ &= (a(c + e) - b(d + f), b(c + e) + a(d + f)) \end{aligned}$$

and $[(a,b) \cdot (c,d)] + [(a,b) \cdot (e,f)]$

$$\begin{aligned} &= (ac - bd, bc + ad) + (ae - bf, af + be) \\ &= ((ac - bd) + (ae - bf), (bc + ad) + (be + af)) \\ &= (a(c + e) - b(d + f), b(c + e) + a(d + f)). \end{aligned}$$

We see that the distributive law holds.

Additive inverse? Since

$$(a,b) + (-a,-b) = (0,0),$$

$(-a,-b)$ is the additive inverse of (a,b).

It is now simple to verify that a non-zero element (a,b) has a multiplicative inverse and hence that the equation

$$(a,b) \cdot (x,y) = (c,d), \qquad (a,b) \neq (0,0)$$

has a unique solution.

Given $(a,b) \neq (0,0)$, we verify that

$$(a,b) \bullet \left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2}\right)$$

$$= \left(a\left(\frac{a}{a^2 + b^2}\right) - b\left(\frac{-b}{a^2 + b^2}\right), a\left(\frac{-b}{a^2 + b^2}\right) + b\left(\frac{a}{a^2 + b^2}\right)\right)$$

$$= (1,0).$$

We now conclude that K together with **+** and **•** *satisfies the conditions imposed on the complex number system.*

At this stage it suffices to redesign our notation for the real numbers in K and to designate the real numbers by the letters, a, b, c,···, to use the standard notations for the additive unit and the multiplicative unit, and to write + and · for **+** and **•** respectively. With these agreements each complex number is of the form

$$a + bi,$$

where a and b are real, and $i^2 = -1$.

Chapter 6

EQUATIONS OF THE FIRST AND SECOND DEGREE IN TWO VARIABLES

6-1. The Straight Line.

In Chapter 2 we took a preliminary look at analytic geometry. The purpose of this chapter is to use the techniques of analytic geometry to study systematically the graphs of equations of the first and second degree in two variables.

One of the axioms of plane geometry is that two distinct points determine a line. In Chapter 2 we defined the slope of the straight line determined by $P_1(x_1,y_1)$ and $P_2(x_2,y_2)$ to be the real number

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

We then used the geometric picture of the straight line to establish the fact that this real number m did not depend on the particular pair of points on the line which were used to compute it. We now use this fact to prove

Theorem 6-1a. If $P_1(x_1,y_1)$ is any point in the plane and m is any real number, then the equation of the straight line passing through the point P_1 with slope m is

6-1a. $$y - y_1 = m(x - x_1).$$

Proof: Let $P(x,y)$ be any point on the line distinct from P_1. Since the slope of the line is independent of the two points used to compute it, regardless of which point $P(x,y)$ on the line we take, so long as it is not P_1 itself, we must have

$$m = \frac{y - y_1}{x - x_1}.$$

$x \neq x_1$ and hence $x - x_1 \neq 0$. (Why can't $x = x_1$?) If we multiply both sides of the equation by $x - x_1$, we have

$$y - y_1 = m(x - x_1).$$

This argument shows that the coordinates of any point on the line, except P_1, satisfy the equation $y - y_1 = m(x - x_1)$. Of course the coordinates of P_1 satisfy the equation also.

There is, however, the possibility that some points on the graph might not lie on the line though P_1 with slope m. For instance, it is conceivable that the graph could be one of the point sets shown below in Fig. 6-1a.

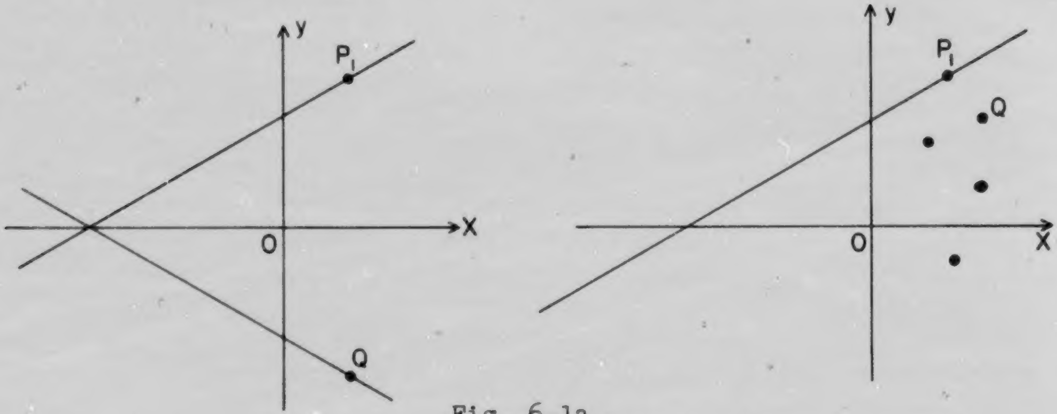


Fig. 6-1a.

Of course our intuition and our experience in plotting points tell us that this is not the case. In order to be absolutely sure, we must prove that every point $Q(x',y')$ whose coordinates satisfy the equation, actually does lie on the line through P_1 with slope m.

Let $Q(x',y')$ be any point whose coordinates satisfy

6-1a $$y - y_1 = m(x - x_1).$$

Then $$y' - y_1 = m(x' - x_1).$$

If $x' = x_1$, then $y' - y_1 = 0$ and $y' = y_1$. This means that Q is just P_1. And since P_1 is certainly on the line, we only need to consider the case $x' \neq x_1$; that is $x' - x_1 \neq 0$. If $x' - x_1 \neq 0$ we can rewrite

$$y' - y_1 = m(x' - x_1) \quad \text{as} \quad \frac{y' - y_1}{x' - x_1} = m.$$

But this equation tells us that the line determined by P_1 and Q has slope m. Q then is on the line through P_1 with slope m. And we have now shown that any point $Q(x',y')$ whose coordinates satisfy the equation 6-1a lies on the line. Since the coordinates of every point on the line satisfies 6-1a and any point whose coordinates satisfy 6-1a lies on the line, equation 6-1a is the equation of the line through P_1 with slope m. The proof is now complete.

The equation $y - y_1 = m(x - x_1)$ is called the point-slope form of the equation of a line.

Example 6-1a. Find the equation of the line passing through the point $(1,2)$ with slope 2.

Solution. By Theorem 6-1a the answer is $y - 2 = 2(x - 1)$. This simplifies to $y = 2x$. To sketch the graph of the equation we simply plot the point $(1,2)$ and use the fact that the slope is 2 to locate a second point on the line, as we did in Chapter 2 . That is, we go to the right 1 and up 2 and find that the point $(2,4)$ is also on the line.

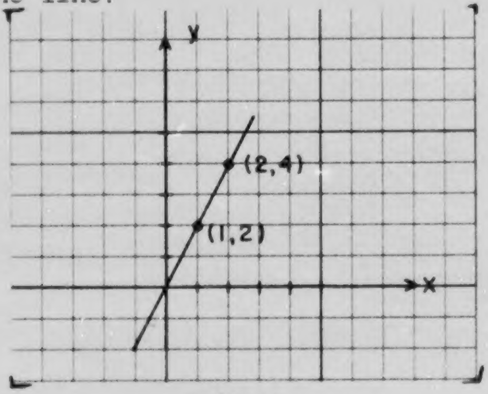


Fig. 6-1b.

An even easier way to plot the line would be to use the point (1,2) which was given and to find one of the intercepts. In our case both the x-intercept and the y-intercept are zero.

If in the point-slope form of the equation of a line, $y - y_1 = m(x - x_1)$, we let $x = 0$, we find the y-intercept to be $y_1 - mx_1$. We set the y-intercept $y_1 - mx_1 = b$ and the equation of the line can be written in the form

6-1b $$y = mx + b.$$

This is an extremely useful form of the equation of a line as both the constants m and b have geometric significance --m gives the slope and b tells us that the line crosses the y-axis at $(0,b)$. For obvious reasons this form of the equation of the line is called the slope-intercept form. It should look familiar to you since it has the same form as the defining equation for the linear function which you met in Chapter 3.

Up to this point we have talked about straight lines which pass through a point and which have slope m. This discussion includes every line which has a slope. However, in Chapter 2 we noted that every non-vertical line has a slope. (What is the slope of a horizontal line?) This means that the only lines which do not have equations which can be written in the slope-intercept form are vertical lines.

Suppose we consider the vertical line through the point (2,0). The point (2,1) is on it. So is the point (2,2) and the point (2,3). In fact all the points with abscissa 2 lie on the line. Furthermore any point which lies on the line has abscissa 2. So the equation $x = 2$ is the equation of the vertical line through the point (2,0).

Similarly every vertical line which crosses the x-axis at $(a,0)$ must have the equation $x = a$. We are now able to assert that every straight line either has the form $y = mx + b$ or $x = a$ for some real numbers m, b, and a.

Example 6-1b. Find the equation of the line through the points (2,-3) and (3,-5).

Solution: In order to use the point-slope form we must find m. The formula for the slope $m = \frac{y_2 - y_1}{x_2 - x_1}$ gives $m = \frac{-3 - (-5)}{2 - 3} = -2$. Substituting in $y - y_1 = m(x - x_1)$ we have $y - (-5) = -2(x - 3)$.

Example 6-1c. Find the equation of the line parallel to $y = -x + 5$ and having x-intercept 2.

Solution: By 6-1b (the slope-intercept form), the line $y = -x + 5$ has slope -1. By Theorem 2 - 3a which says that parallel lines have the same slope, the slope of the line we are after is -1. We use the point-slope form with $m = -1$ and (2,0) as the point $P_1(x_1,y_1)$ to obtain the equation $y - 0 = -(x - 2)$.

Exercises 6-1

1. Write an equation of the line which passes through the two points:
   (a) $P_1(2,4)$ and $P_2(4,5)$
   (b) $P_1(2,4)$ and $P_2(4,2)$
   (c) $P_1(0,0)$ and $P_2(1,5)$
   (d) $P_1(10,2)$ and $P_2(0,0)$
   (e) $P_1(2,7)$ and $P_2(-8,5)$
2. Draw the graph of each of the following equations on the same set of coordinate axes:
   (a) $y = 2x + 1$
   (b) $y = 4x + 1$
   (c) $y = -3x + 1$
   (d) $y = -x + 1$

3. Draw the graphs of $y = 2x - 3$, $y = 2x$ and $y = 2x + 4$ on the same set of coordinate axes.

4. (a) Draw lines through point $(2,2)$ having $m = -2, -1, 0, 1, 2$.

   (b) Write an equation of each line.

5. Write an equation of the vertical line passing through $(3,4)$.

6. Determine m so that the line whose equation is $y = mx + 3$ passes through the point $(-1,-2)$.

7. Write an equation of the line with slope $-\frac{1}{2}$ and x-intercept 2.

8. Write an equation of the line passing through the origin and the point $(-1,3)$.

9. Write an equation of the line passing through the origin and the point (x_1,y_1).

10. Write an equation of the line passing through the origin with slope m. In many practical problems this relation between two variables x and y is called direct variation. If $y = kx$ we say that y varies directly as x, or that y is proportional to x. In the latter case, k is called the constant of proportionality.

11. Write an equation expressing the relation between variables of the following:

    (a) The perimeter of an equilateral triangle varies directly as the length of a side.

    (b) The number of feet s traversed by a freely falling body varies directly as the square of its time of fall t.

    (c) The current I in an electric circuit varies directly as the electromotive force E.

12. If x varies as y, and $x = 8$ when $y = 15$, find x when $y = 10$.

13. The volume V of an ideal gas varies directly as its absolute temperature T. If $V = 1500$ cc. when $T = 300^\circ$ absolute what will the temperature be when volume is 2500 cc?

14. Find the value of k for which the line $y = kx + k$ will pass through the point $(-\frac{1}{2},-3)$.

15. Write an equation of the line passing through $(3,4)$ parallel to the line whose equation is $y = 2x + 2$.

16. Write an equation of the line through the origin perpendicular to the line whose equation is $y = \frac{1}{4}x + \frac{1}{2}$.

17. Write an equation of a line through the point $(-2,5)$ and perpendicular to $5x - 2y = 2$.

18. A line has a slope $\frac{3}{4}$ and passes through the point $(8,-12)$. Write an equation of a second line through this point perpendicular to the first line.

19. Graph the lines on the same coordinate axes whose equation is $5x + 3y - c = 0$ and having y-intercepts

    (a) $-\frac{5}{2}$ (d) 5

    (b) -1 (e) $\frac{7}{4}$

    (c) 0

20. Write an equation of the line which passes through the point $(-5,7)$ and is parallel to,

    (a) the y-axis (b) the x-axis

21. Write an equation of the line perpendicular to the line whose equation is $2y + x = 5$ and intersecting it, on,

    (a) the y-axis (b) the x-axis

22. Write an equation of the line tangent at the point $(3,4)$ to the circle with center at the origin and radius 5.

23. Find an equation of the line at a distance 4 from the origin with x-intercept 5.

24. Find an equation of the line which passes through the origin and the midpoint of the segment cut off by the coordinate axes on the line whose equation is $2x + 3y + c = 0$.

25. Find an equation of the line which contains the shortest line segment that joins the origin and a point on the line whose equation is $y - 2x = 10$.

6-2. The General Linear Equation $Ax + By + C = 0$.

Definition 6-2a. The equation

6-2a $$Ax + By + C = 0,\ A^2 + B^2 \neq 0$$

is called the general linear equation in two variables x and y. ($A^2 + B^2 \neq 0$ is an economical way of saying either A or B is not zero.)

In the last section we showed that the graph of the equations $y = mx + b$ and $x = a$ are straight lines. We now ask: Is the graph of every linear equation a straight line? And conversely, Is every straight line the graph of some linear equation? The answer is given by

Theorem 6-2a: The graph of every linear equation is a straight line and every straight line is the graph of a linear equation.

Proof: Every linear equation has the form $Ax + By + C = 0$, $A^2 + B^2 \neq 0$. If $B \neq 0$,

$$y = -\frac{A}{B}x - \frac{C}{B},$$

which is the equation of the line with slope $m = -\frac{A}{B}$ and y-intercept $b = -\frac{C}{B}$ by Theorem 6-1a.

If $B = 0$, then $A \neq 0$ and $x = -\frac{C}{A}$, which is the equation of the vertical line through the point $(-\frac{C}{A}, 0)$. Therefore every linear equation has for its graph a straight line.

Conversely, every straight line is either vertical and has an equation $x = a$ for some real number a, or the line has a slope m and a y-intercept b, and has an equation of the form $y = mx + b$. Since both of these equations can be written in the form of the general linear equation -- $(1)x + (0)y + (-a) = 0$ and $(-m)x + (1)y + (-b) = 0$, the theorem is proved

Example 6-2a. What is the slope of the line whose equation is $3x + 2y + 7 = 0$?

Solution: The given equation can be written in the form $y = -\frac{3}{2}x - \frac{7}{2}$. Hence the slope of the line is $-\frac{3}{2}$.

In this example we see that a line may be the graph of different equations. Thus, the equations $3x + 2y + 7 = 0$ and $y = -\frac{3}{2}x - \frac{7}{2}$ are equations of the same line.

For two equations whose graphs are vertical lines, it is easy to see whether or not the two equations are equations for the same line. Since in this case $B = 0$, the equations have the form $Ax + C = 0$ or $x = -\frac{C}{A}$. For example $2x = 3$ and $4x = 6$ are equations for the same straight line; namely, the vertical line with x-intercept $\frac{3}{2}$.

For equations whose graphs are non-vertical lines, we simply write them in slope-intercept form and compare slopes and intercepts. For example $2x + y - 4 = 0$ and $4x + 2y - 6 = 0$ are not equations of the same line since their slope-intercept forms are $y = -2x + 4$ and $y = -2x + 3$. From this form of the equations we know that the lines have the same slope, -2, and are therefore parallel. But their y-intercepts are 4 and 3 respectively, and hence they certainly are not the same line.

On the other hand $Ax + By + C = 0$ and $kAx + kBy + kC = 0$ both have the slope intercept form $y = -\frac{A}{B}x - \frac{C}{B}$ and are equations for the same line. The result can be stated in the following way: The graphs of two linear equations of the form $Ax + By + C = 0$ are the same line if and only if their corresponding coefficients are proportional.

Example 6-2b. Determine without drawing graphs whether the following pairs of equations have as their graphs lines which are the same or are parallel:

(a) $5x + 10y - 25 = 0,$ $\quad$ (b) $x = 2y - 3,$
$-x - 2y + 5 = 0;$ $\quad$ $6y = 3x - 2.$

Solution: (a) $\frac{5}{-1} = \frac{10}{-2} = \frac{-25}{5} = -5$, the two equations represent the same line.

(b) If we first rewrite the equations in the form $Ax + By + C = 0$, we have $x - 2y + 3 = 0$, $-3x + 6y + 2 = 0$. Then $\frac{1}{-3} = \frac{-2}{6} \neq \frac{3}{2}$; hence the equations do not represent the same line. However, since the slopes of both lines are $\frac{1}{2}$, the lines are parallel.

We have now obtained several forms for equations of straight lines -- the point-slope form, the slope-intercept form, and the general form. The first is convenient if the line is given by a point and the slope. The second allows us to read off the slope and the y-intercept. However, only non-vertical lines can be written in these two forms, whereas, an equation for any line can be written in the general form. Another useful form is given in the following example.

Example 6-2c. Find the equation of the line whose x-intercept is a and whose y-intercept is b, where $a \neq 0$ and $b \neq 0$.

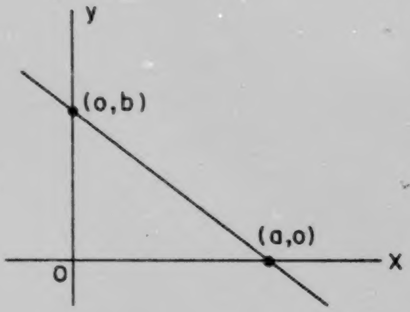


Fig. 6-2

Solution: The slope of the line is

$$m = \frac{b - 0}{0 - a} = -\frac{b}{a},$$

and it crosses the y-axis at the point $(0,b)$. Therefore its equation is

$y = -\frac{b}{a}x + b$, or equivalently

6-2b $\frac{x}{a} + \frac{y}{b} = 1.$

Equation 6-2b is called the intercept form of the equation of a straight line.

Exercises 6-2

1. Write an equation of the line that has slope $\frac{3}{5}$ and passes through point $(-1,-2)$. Write it in the form $Ax + By + C = 0$
2. Use the intercept form to write an equation of the line having x-intercept 2 and y-intercept 3.
3. Find the slope and the y-intercept of the lines whose equations are;

   (a) $3x - 2y - 6 = 0$

   (b) $x - 8y + 2 = 0$

   (c) $5y - 9x - 1 = 0$

   (d) $4x - y - 7 = 0$

   (e) $8x - 2y - 7 = 0$

   (f) $-x + \frac{1}{4}y + 7 = 0$

4. Find the x and y-intercepts of the lines whose equations are given, by first writing each equation in the intercept form:

(a) $3x + 2y - 6 = 0$

(b) $4x - 3y - 12 = 0$

(c) $5x + 2y - 10 = 0$

(d) $4x - 7y - 20 = 0$

(e) $3x - 5y + 10 = 0$

(f) $2x - 3y + 5 = 0$

5. Consider the following pairs of equations. Without sketching graphs, determine which pairs represent lines which are the same, are parallel, or are neither.

(a) $3x - 2y - 2 = 0$
$6x - 4y - 4 = 0$

(b) $2x - 2y + 7 = 0$
$3x - 6y + 1 = 0$

(c) $x - y - 6 = 0$
$x + y - 6 = 0$

(d) $6x + 2y + 5 = 0$
$x + 3y + 5 = 0$

(e) $6y = x - 3$
$-3\frac{1}{2}x + 21y = -2$

(f) $3x + y - 1 = 0$
$2x + \frac{2y}{3} = \frac{2}{3}$

(g) $2x + 1 - y = 0$
$x - \frac{5}{2} - \frac{y}{2} = 0$

6. Write an equation of a line which passes through the point $(0,0)$ and is parallel to the line whose equation is $2x - y - 5 = 0$.

7. Write an equation of a line which passes through the point $(-2, \frac{1}{2})$ and is perpendicular to the line whose equation is $\frac{x}{5} - \frac{y}{6} = 1$.

6-3. The Parabola.

The first two sections of this chapter were concerned with the first degree equation and its graph, the straight line. We proved that every straight line in the xy-plane is the graph of a first degree equation in the variables x and y, and conversely. We showed in plane geometry that the set of points equidistant from two fixed points is a straight line (perpendicular bisector of the segment joining the two points). A natural question to ask next is, what is the set of points equidistant from a point and a line? The answer is given by the following definiton.

Definition 6-3a. The set of points equidistant from a line and a point off the line is called a parabola. The line is called the directrix, and the point is called the focus. The line through the focus perpendicular to the directrix is called the axis, or, sometimes the axis of symmetry, of the parabola.

In Fig. 6-3a, DD' is the directrix and F is the focus. The intersection of the axis of the parabola with the directrix is the point R, and the midpoint of RF is V. The point V is on the parabola because $d(R,V) = d(V,F)$. The point V is called vertex of the parabola.

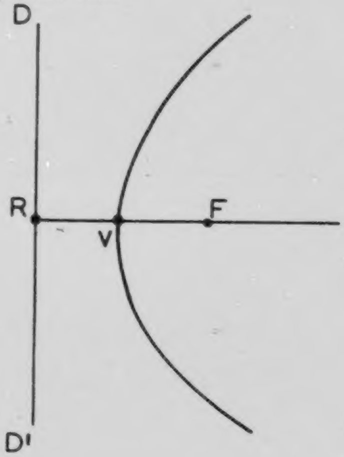


Fig. 6-3a

Example 6-3a. Find the equation of the parabola which has the directrix $x = -2$ and focus $(2,0)$.

Solution: Our problem is to find an equation which is satisfied by the set of points (x,y) which are equidistant from the line $x = -2$ and the point $(2,0)$. In Fig. 6-3b, let $P(x,y)$ be any such point, and let Q be the intersection of the perpendicular from P to the line $x = -2$ with that line. Then since PQ is horizontal, Q has coordinates $(-2,y)$. Since P is equidistant from $F(2,0)$ and the line $x = -2$,

$$d(P,F) = d(P,Q)$$

$$\sqrt{(x-2)^2 + (y-0)^2} = \sqrt{(x+2)^2 + (y-y)^2}$$

$$x^2 - 4x + 4 + y^2 = x^2 + 4x + 4$$

$$y^2 = 8x.$$

Up to this point we have shown that the coordinates of any point equidistant from the point $(2,0)$ and the line $x = -2$ satisfy the equation $y^2 = 8x$. Conversely, it may be shown (See Problem 5, Exercises 6-3) that if the coordinates of a point satisfy the equation $y^2 = 8x$, then the point is equidistant from the line $x = -2$ and the point $(2,0)$. Thus $y^2 = 8x$ is the equation of the parabola.

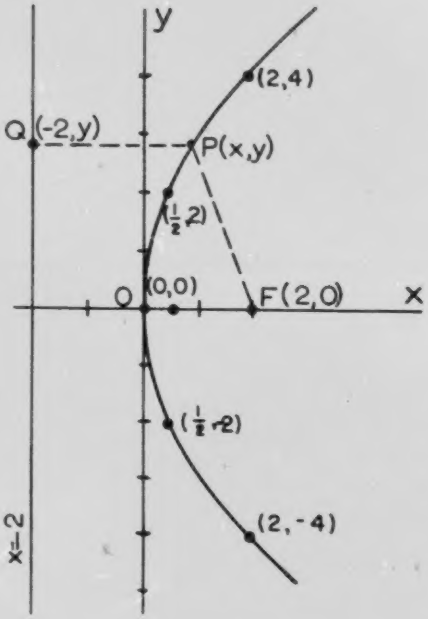


Fig. 6-3b

To sketch the parabola we use the techniques of Chapter 2. The x and y intercepts are both 0. Furthermore since $(-y)^2 = y^2 = 8x$, the curve is symmetric with respect to the x-axis. Since $x = \frac{y^2}{8} \geq 0$, there are no points on the graph for $x < 0$. We plot a few convenient points and draw a smooth curve through them.

x	0	$\frac{1}{2}$	2
y	0	± 2	± 4

These points are shown in Fig. 6-3b.

Example 6-3b. Find the equation of the set of points equidistant from the line $y = -3$ and the point $(0,3)$; that is, the parabola with directrix $y = -3$ and focus $(0,3)$. As in Example 6-3a we begin with

$$d(P,F) = d(P,Q)$$

$$\sqrt{(x-0)^2+(y-3)^2} = \sqrt{(x-x)^2 + (y+3)^2}$$

This simplifies to

$$x^2 = 12y.$$

If we test for symmetry, $(-x)^2 = x^2 = 12y$, and the parabola is symmetric with respect to the y-axis. The vertex is at the origin. This information together with the points obtained from the following table enables us to sketch the parabola as shown in Fig. 6-3c.

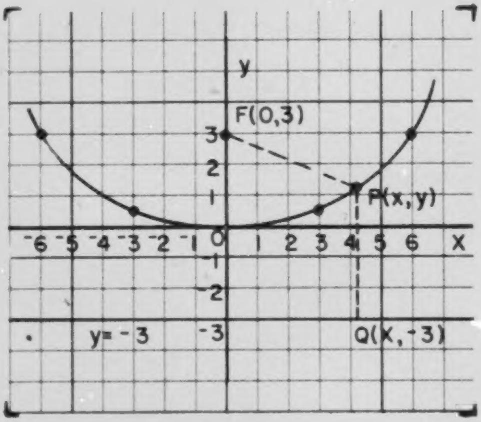


Fig. 6-3c

x	-6	-3	0	3	6
y	3	$\frac{3}{4}$	0	$\frac{3}{4}$	3

We now consider the more general problem of finding the equation of the parabola with focus $F(0,c)$ and directrix the line $y = -c$. As before we let $P(x,y)$ be any point on the parabola. The $Q(x,-c)$ is the foot of the perpendicular from P to the directrix. See Fig. 6-3d. Then

$$d(P,F) = d(P,Q)$$

or $\sqrt{(x-0)^2 + (y-c)^2} = \sqrt{(x-x)^2 + \left(y-(-c)\right)^2}$

$$x^2 + (y-c)^2 = (y+c)^2$$

6-3a $\boxed{x^2 = 4cy.}$

The vertex is at the origin; the parabola is symmetric with respect to the y-axis, which is the axis of the parabola.

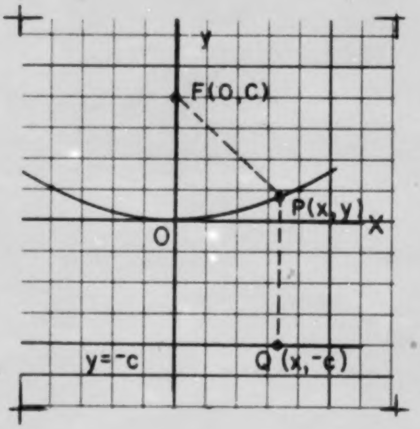


Fig. 6-3d.

If we had taken the directrix to be the line $x = -c$ and the focus to be $(c,0)$, a similar agrument would have given the equation

6-3b $\boxed{y^2 = 4cx}$

These two equations are sometimes referred to as the standard forms of the equation for the parabola. In these forms the vertex is at the origin and the absolute value of the constant c is the distance of the focus and the directrix from the origin.

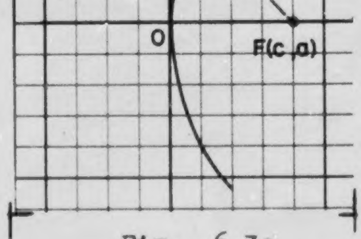


Fig. 6-3e

If we consider the more general case in which the focus is any point (a,b) and the directrix is any line parallel to one of the coordinate axes, only the algebra is more difficult.

Example 6-3c. Find the equation of the parabola with focus (4,2) and directrix the line x= -6.

Solution: Let P(x,y) be any point on the parabola and let Q(-6,y) be the point in which the perpendicular from P to the directrix meets the directrix.

$$d(P,F) = d(P,Q)$$

$$\sqrt{(x-4)^2 + (y-2)^2}$$

$$= \sqrt{(x-(-6))^2 + (y-y)^2}$$

$$(x-4)^2 + (y-2)^2 = (x + 6)^2$$

$$(y-2)^2 = 12x + 8x + 36 - 16$$

$$(y-2)^2 = 20(x + 1)$$

$$(y-2)^2 = 20\left(x - 1\ (-1)\right).$$

The vertex is the point V(-1,2).

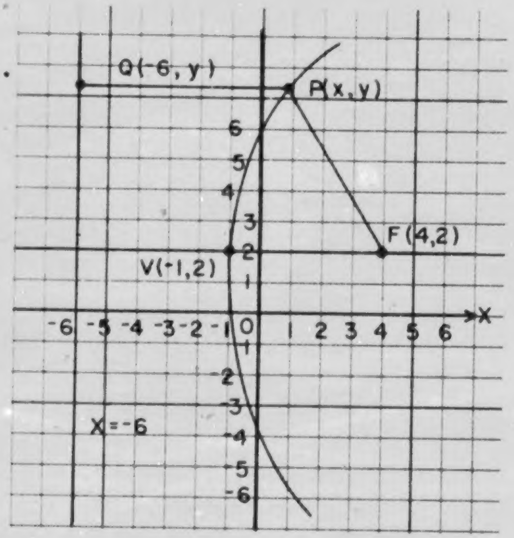


Fig. 6-3f

In general, if the equation has the form

6-3c

$$\left\{\begin{array}{l}(y - k)^2 = 4c(x - h)\\ (x - h)^2 = 4c(y - k)\end{array}\right| \begin{array}{l}\text{it is a parabola}\\ \text{with vertex } V(h,k),\end{array}$$

$$\text{focus} \left\{\begin{array}{c}(h + c,k)\\ \\ (h,k + c)\end{array}\right\} \text{ directrix } \left\{\begin{array}{c}x = h - c\\ \\ y = k - c\end{array}\right\}$$

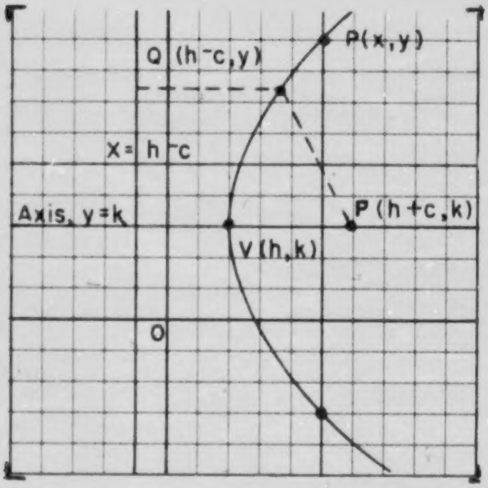

P(x,y)
Q (h−c, y)
X = h−c
Axis, y=k
P (h+c, k)
V (h, k)
O

Fig. 6-3g

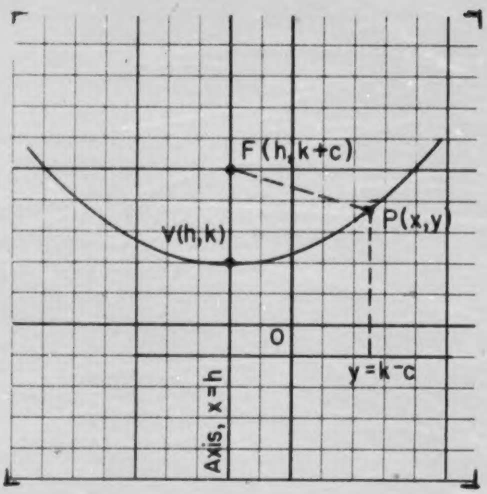

F (h, k+c)
V(h,k)
P(x,y)
O
y = k−c
Axis, x=h

Fig. 6-3h

Example 6-3d. Find the coordinates of the vertex, the focus, and the equation of the directrix of the parabola:

(a) $x^2 + 6x - 2y + 3 = 0$,

(b) $y^2 + 4x + 8y + 4 = 0$.

Solution: Using the method of completing the square (Chapter 3),

(a) $$\begin{aligned} x^2 + 6x + 9 &= 2y - 3 + 9 \\ (x+3)^2 &= 2y + 6 \\ (x+3)^2 &= 2(y+3) \\ (x+3)^2 &= 4(\tfrac{1}{2})(y+3). \end{aligned}$$

By 6-3c, $h = -3$, $k = -3$, $c = \frac{1}{2}$.

Hence $V(-3,-3)$; $F(-3,-\frac{5}{2})$; DD': $y = -\frac{7}{2}$.

(b) $$\begin{aligned} y^2 + 4x + 8y + 4 &= 0 \\ y^2 + 8y + 16 &= -4x - 4 + 16 \\ (y+4)^2 &= -4(x-3) \\ (y+4)^2 &= 4(-1)(x-3) \end{aligned}$$

By 6-3c, $h = 3$, $k = -4$, $c = -1$.

Hence $V(3,-4)$; $F(2,-4)$; DD': $x = 4$.

The parabola has the interesting and useful physical property that a ray of light emanating from the focus will be reflected from a parabolic surface in a line parallel to its axis. This property is the reason for the parabolic shape of automobile headlights and the metal reflectors in flashlights. The reflected light is then concentrated in a beam which can be directed where it will be most useful.

Exercises for 6-3.

1. Find an equation of the parabola and sketch the graph showing the focus and the directrix of each:

(a) directrix $x = -3$ and focus $(3,0)$.

(b) directrix $x = 4$ and focus $(-4,0)$.

(c) directrix $y = 5$ and focus $(0,-5)$.

(d) directrix $y = -6$ and focus $(0,6)$.

2. Find the coordinates of the focus, the equation of the directrix, and sketch the graph of each of the following:

   (a) $x^2 = -4y$

   (b) $x^2 = 4y$

   (c) $y^2 = -6x$

   (d) $x^2 = -6y$

   (e) $x = y^2$

   (f) $x^2 + y = 0$

   (g) $2x^2 - 4y = 0$

   (h) $3x + 4y^2 = 0$

3. Give several examples of a parabola from the physical world.

4. For each of the following parabolas find an equation of its axis, its directrix, and the coordinates of its vertex and its focus. Sketch the curve.

   (a) $y = \frac{1}{12}x^2$

   (b) $y = -\frac{1}{16}x^2$

   (c) $y^2 = 20x$

   (d) $x = -2y^2$

   (e) $x + y^2 = 0$

   (f) $x^2 - y = 0$

5. Complete the proof of Example 6-3a. That is, prove that if a point (x,y) has coordinates which satisfy the equation $y^2 = 8x$, then the point is equidistant from the line $x = -2$ and the point $(2,0)$. (HINT: Try to read the proof backwards and supply reasons for each step.)

6. The area of a circle varies directly as the square of the radius.

   (a) What is the constant of proportionality?

   (b) Write an equation.

   (c) Sketch the graph.

   (d) If the measure of the area of a circle is 63, find its diameter.

7. Sketch the graph of,

(a) $x = +\sqrt{y}$ (c) $y = +\sqrt{x}$

(b) $x = -\sqrt{y}$ (d) $y = -\sqrt{x}$

Is each a parabola? Discuss.

8. Find an equation for each parabola having the following foci and directrices and sketch:

(a) Focus $(0,2)$ and directrix the x-axis.

(b) Focus $(0,-2)$ and directrix the x-axis.

(c) Focus $(0,2)$ and directrix $y = -4$.

(d) Focus $(2,0)$ and directrix the y-axis.

(e) Focus $(-2,0)$ and directrix the y-axis.

(f) Focus $(-2,0)$ and directrix $x = 1$.

(g) Focus $(1,2)$ and directrix $x = -2$.

(h) Focus $(2,-1)$ and directrix $x = 4$.

(i) Focus $(-1,2)$ and directrix $y = -3$.

(j) Focus $(1,-2)$ and directrix $y = 2$.

(k) Focus $(2a,0)$ and directrix $x = a$, $a > 0$.

(l) Focus $(2a,a)$ and directrix $x = a$, $a > 0$.

9. Given the equation $x^2 - 4y + 16 = 0$

(a) Sketch the graph.

(b) Where does the line whose equation is $y - 8 = 0$ intersect the curve?

(c) Describe the intersection of the line whose equation is $y - 3 = 0$ with the curve.

10. For each of the following parabolas find the coordinates of the vertices, an equation of the axis of symmetry, and sketch the curve:

    (a) $y^2 + 2y - 5x + 11 = 0$

    (b) $x^2 - 2x - y + 8 = 0$

    (c) $2y^2 + 28y - x + 101 = 0$

    (d) $3y^2 - 24y - x + 47 = 0$

    (e) $140y^2 + 140y - 80x - 20 = 0$

    (f) $4a^2y^2 + 8a^3y - x + 4a^4 + a = 0$ $(a > 0)$.

11. A line segment perpendicular to the axis of the parabola at its focus whose end points are on the parabola is called the latus rectum. Show that the length of the latus rectum of a parabola is two times the distance between the directrix and the focus. Note: The latus rectum is also called the "focal chord", see page 359.

12. Find the length of the latus rectum of the parabolas whose equations are,

    (a) $y^2 = x$

    (b) $x^2 = y$

    (c) $y^2 = 4x$

    (d) $y = \frac{1}{12}x^2$

    (e) $x^2 = -6y$

    (f) $-3x = y^2$

13. Find an equation of the parabola whose latus rectum equals 4, vertex is at the origin and the axis is the x-axis.

14. Write an equation of the parabola whose focus is $(2,-3)$ and vertex is $(1,-3)$.

15. Write an equation of the parabola whose vertex is at the origin, axis is the x-axis, and passing through the point $(-3,-2)$. What is the focus?

16. Write an equation of the parabola passing through $(-3, +5)$ and the origin and having as its axis of symmetry the y-axis. What is the focus of the parabola?

17. Write an equation of the parabola having the end points of its latus rectum at $(4,6)$ and $(4,-8)$, and its vertex at the origin.

18. Write an equation of the parabola whose focus is $(0,-2)$ its directrix is parallel to the x-axis, and the length of its latus rectum equal 8.

19. Find the value of a so that the parabola whose equation is $y = ax^2$ will pass through,

    (a) The point whose coordinates are $(3,18)$.

    (b) The point whose coordinates are (x_0,y_0).

    Can this be done for any point?

20. Consider the parabola whose equation is $y = x^2 + x + 5$. Replace the x by $x - 2$.

    (a) Write the "new" equation.

    (b) Sketch the graphs of each of these equations on the same coordinate axes.

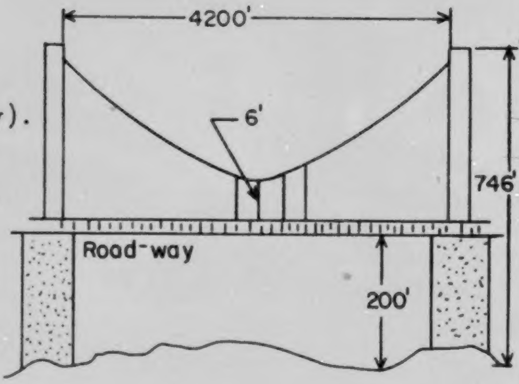
    (c) Replace x in the equation $y = x^2 + x + 5$ by $x + 2$, write the "new" equation and sketch its graph on the same set of coordinate axes as for (b) above.

    (d) Discuss anything interesting which you observe about these curves.

21. A comet moves in a parabolic orbit with the sun at the focus. When the comet is 4×10^7 miles from the sun, the line from the sun to it makes an angle of 60° with the axis of the orbit (drawn in the direction in which the orbit opens). Find how near the comet comes to the sun.

22. The longitudinal section of a reflector is a parabola 16 inches across and 8 inches deep. How far from the vertex is the focus?

23. A cable of the Golden Gate suspension bridge is in the shape of a parabola (ideally). The supporting towers of the cable are 4,200 feet apart. The cable passes over the supporting towers 746 feet above the bay. The bridge is 200 feet above the bay. The lowest point of the cable is 6 feet above the road-way. Find the lengths of supporting rods (from the cable to the road-way) at 100-foot intervals from the center of the bridge to one of the towers.



6-4 The General Definition of the Conic

In this Chapter we have considered the set of points equidistant from two fixed points and the set of points equidistant from a fixed point and a fixed line. We now carry this process one step further. We do not demand that the point be the same distance from the fixed point and the fixed line, but that the distance from the point be some constant times the distance from the line.

Example 6-4a. Find an equation of the set of points with the property that the distance of each point from the point $(1,0)$ is one-half the distance from the line $x = 4$.

Solution: See Fig. 6-4a.

$$d(F,P) = \frac{1}{2}d(P,Q)$$

$$\sqrt{(x-1)^2 + y^2}$$

$$= \frac{1}{2}\sqrt{(x-4)^2 + (y-y)^2}$$

$$(x-1)^2 + y^2 = \frac{1}{4}(x-4)^2$$

$$x^2 - 2x + 1 + y^2 = \frac{1}{4}x^2 - 2x + 4$$

$$\frac{3}{4}x^2 + y^2 = 3$$

$$\frac{x^2}{4} + \frac{y^2}{3} = 1.$$

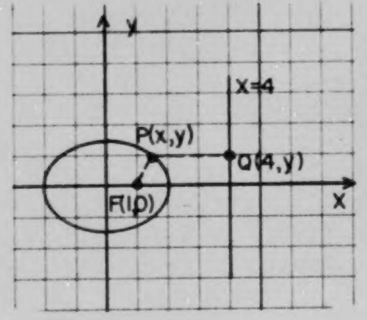


Fig. 6-4a

Example 6-4b. Find an equation of the set of points, each of which is twice as far from (4,0) as from the line x = 1.

Solution: See Fig 6-4b

$$d(F,P) = 2d(P,Q)$$

$$\sqrt{(x-4)^2 + y^2}$$

$$= 2\sqrt{(x-1)^2 + (y-y)^2}$$

$$(x-4)^2 + y^2 = 4(x-1)^2$$

$$x^2 - 8x + 16 + y^2 = 4x^2 - 8x + 4$$

$$-3x^2 + y^2 = -12$$

$$\frac{x^2}{4} - \frac{y^2}{12} = 1$$

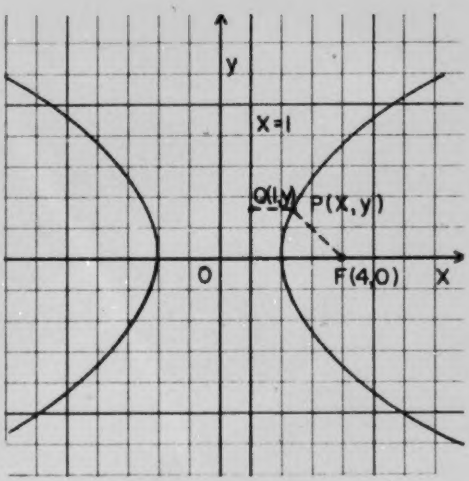


Fig. 6-4b

We adopt a notation similar to that of Section 6-3 and call the fixed point the focus and designate it by $F(c,0)$; the fixed line, the directrix, and let it have the equation $x = d$; the constant, the eccentricity, and denote it by the letter e. Then (See Fig. 6-4c)

$$d(P,F) = e \cdot d(P,Q)$$

$$\sqrt{(x - c)^2 + y^2} = e\sqrt{(x - d)^2 + (y - y)^2}$$

$$(x - c)^2 + y^2 = e^2(x - d)^2$$

6-4a $$x^2(1-e^2) + 2x(de^2-c) + y^2 = e^2d^2 - c^2$$

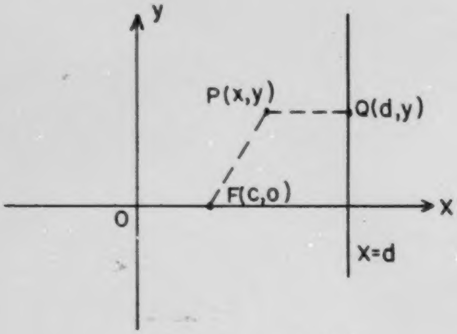


Fig. 6-4c

If we take $e = 1$ and $d = -c$, we see that we get the equation $y^2 = 4cx$, which was the equation of the parabola, (6-3b). Since this case has been discussed in great detail, we now concentrate on the cases in which e is positive but not equal to 1.

In order to simplify the equation 6 - 4a, we choose the constant d to be $\frac{c}{e^2}$, making the coefficient of x zero. Geometrically this simply determines the position of the directrix. The equation becomes

6-4b $$x^2(1 - e^2) + y^2 = \frac{c^2}{e^2}(1 - e^2).$$

The tests for symmetry (Chapter 2) tell us that the graph of the equation is symmetric with respect to both of the coordinate axes and the origin. The x-intercepts are $\pm \frac{c}{e}$; the y-intercepts are $\pm \frac{c}{e}\sqrt{1 - e^2}$. But if $e > 1$, $\sqrt{1 - e^2}$ is not real and there are no intercepts. We therefore consider two cases.

Case 1: $e < 1$. We use the same notation for intercepts which we used for the straight line and let $\frac{c}{e} = a$ and $\frac{c}{e}\sqrt{1 - e^2} = b$. (We have tacitly assumed that c and e and therefore a and b are positive. This will be understood in all that follows.) Then

$b = \frac{c}{e}\sqrt{1 - e^2} = a\sqrt{1 - e^2}$, and $\frac{b}{a} = \sqrt{1 - e^2}$. If we now substitute $\frac{b^2}{a^2}$ for $(1 - e^2)$ and b^2 for $\frac{c^2}{e^2}(1 - e^2)$ in equation 6-4a, we have

$$\frac{x^2b^2}{a^2} + y^2 = b^2,$$

or

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ (e < 1).}$$

Case 2: $e > 1$. We multiply both sides of equation 6 - 4b by -1 and the equation becomes

$$x^2(e^2 - 1) - y^2 = (e^2 - 1)\frac{c^2}{e^2},$$

and if we let $\frac{c}{e} = a$ and $\frac{c}{e}\sqrt{e^2 - 1} = b$, the equation becomes

$$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \ (e > 1).}$$

These two cases lead us to make the following definitions:

Definition 6-4a. The set of points P with the property that the distance from P to a fixed point is equal to a constant, e, $0 < e < 1$, times the distance from P to a fixed line, is called an *ellipse*. The fixed point is called the *focus*. The fixed line is called the *directrix*. The constant e is called the *eccentricity*.

Definition 6-4b. The set of points P with the property that the distance from P to a fixed point is equal to a constant, $e > 1$, times the distance from P to a fixed line, is called a *hyperbola*. The fixed point is called the *focus*. The fixed line is called the *directrix*. The constant e is called the *eccentricity*.

We may summarize these definitions and the definition of the parabola (6-3a) in the following table:

THE CONIC SECTIONS

If $e = 1$, the conic is a *parabola*.
If $e < 1$, the conic is an *ellipse*.
If $e > 1$, the conic is a *hyperbola*.

These curves -- the parabola, the ellipse, and the hyperbola -- are called *conic sections*, since all of them can be obtained as plane sections of a right circular cone. In addition to these curves, one can also obtain a circle, a straight line, and two intersecting straight lines as special cases of plane sections of a cone.

The equations which we derived for all of these curves are equations of the second degree in x and y. This is not coincidental. It can be shown that every equation of the second degree in x and y,

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0,$$

has for its graph a conic section (or one of the limiting forms of these curves mentioned above). See Problems 8 and 9 in Exercise 6-4. Conversely, every conic section (no matter what the position of the focus and the directrix) is the graph of an equation of the second degree in x and y.

These statements tell the whole story for second degree equations in two variables. Our study of analytic geometry has now furnished us with a complete description of all graphs of first and second degree equations -- they are simply straight lines and the conic sections (or limiting forms of these curves). We will study in more detail the properties of the circle, ellipse, and hyperbola in the next two sections.

Exercises 6-4

1. Find an equation of the set of points with the property that the distance of each point from $P(2,0)$ is $\frac{1}{3}$ the distance from the line whose equation is $y = 3$. Identify the conic.

2. The focus of a conic is the origin and the corresponding directrix is the line whose equation is $y = -2$. The eccentricity is $\frac{3}{2}$.

   (a) Identify the conic.

   (b) Write an equation of the curve.

3. The eccentricity of a conic is 1. The focus is the point $F(-2,3)$ and the directrix is the line whose equation is $x = 4$.

   (a) Identify the conic.

   (b) Write an equation of the conic.

4. The eccentricity of a conic is $\sqrt{2}$, the focus is $(-3,0)$, and the directrix is $3x - 2 = 0$.

   (a) Identify the conic.

   (b) Write an equation of the conic.

5. The focus of a conic is $(-1,3)$, the directrix is $2x - 1 = 0$ and the eccentricity is $2\sqrt{5}$.

   (a) Identify the curve.

   (b) Write an equation of the conic.

6. Write an equation for each set of data.

	Focus	Directrix	e
(a)	(-2,3)	$y = -2$	2
(b)	(1,1)	$x = 2$	$\frac{1}{2}$
(c)	(1,-2)	$y = \frac{2}{5}$	$\frac{5}{2}$
(d)	(-1,-3)	$x = 0$	1
(e)	(3,-5)	$x = 0$	$\frac{2}{3}$

* Sketch the graph of each.

7. Identify the conic and sketch the graph of each of the following:

(a) $2x^2 + 3y^2 = 6$

(b) $4x^2 - 16y^2 = 16$

(c) $4x^2 + 16y^2 = 16$

(d) $4x^2 + 16y = 0$

(e) $9x^2 + 9y^2 = 4$

(f) $y^2 = 9x - 36$

*8. Discuss the conic of the equation,

$$Ax^2 + Cy^2 + F = 0$$

(a) If $A \cdot C > 0$.

(b) If $A \cdot C < 0$.

(c) If $A \cdot C = 0$.

*9. Discuss the conic of the equation,

$$Ax^2 + Cy^2 + DX + Ey + F = 0$$

(a) If $A \cdot C > 0$.

(b) If $A \cdot C < 0$.

(c) If $A \cdot C = 0$.

*10. The eccentricity of a conic is $\frac{2}{3}$; its focus is the point whose coordinates are $F(2,-1)$; its directrix is the line whose equation is $y = x$.

(a) Identify the conic.

(b) Write an equation of the conic.

6-5. The Circle and the Ellipse.

We begin this section by reviewing the derivation of the equation for the circle, which we met in Chapter 2.

Let $C(h,k)$ be a point in the plane and r be a positive real number. *The circle with center C and radius r* is the set of all points $P(x,y)$ such that the distance from P to C is equal to r. Then (see Fig. 6-5a)

$$d(C,P) = r$$

$$\sqrt{(x - h)^2 + (y - k)^2} = r$$

6-5a $$\boxed{(x - h)^2 + (y - k)^2 = r^2}\ .$$

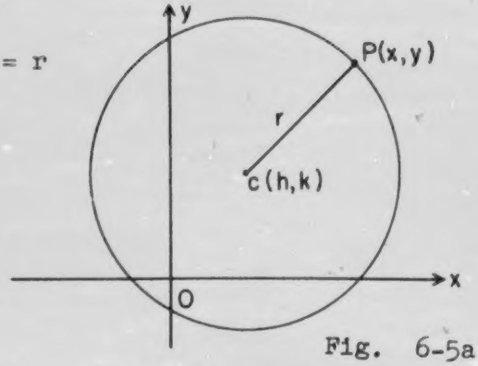


Fig. 6-5a

We have shown that every point on the circle must satisfy equation 6-5a. Conversely, if the coordinates x and y of any point P satisfy 6-5a, then the point lies on the circle. For since r is positive, taking the positive square root of both sides of 6-5a, we have

$$\sqrt{(x - h)^2 + (y - k)^2} = r,$$

$$d(C,P) = r,$$

and therefore P lies on the circle with radius r and center at $C(h,k)$.

We have proved that the coordinates of every point on the circle satisfy equation 6-5a and conversely that every point whose coordinates satisfy 6-5a lie on the circle. Therefore, equation 6-5a is the equation of the circle with center $C(h,k)$ and radius r.

If we consider the special case in which the center is at the origin, the equation assumes the simpler form

6-5b $$\boxed{x^2 + y^2 = r^2.}$$

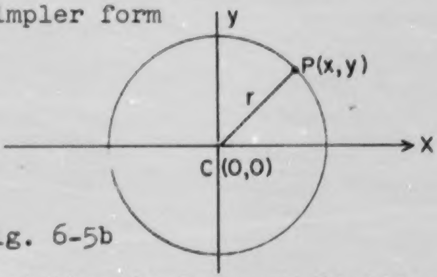


Fig. 6-5b

We said in the last section that the circle was a limiting case of the ellipse. Let us now turn, then, to the ellipse. We shall show presently how the circle can be obtained from the ellipse.

The ellipse was defined as the conic with eccentricity $0 < e < 1$. We recall that if we take the focus to be $F(c,0)$, the directrix to be the line $x = \frac{c}{e^2}$, and let $a = \frac{c}{e}$ and $b = \frac{c}{e}\sqrt{1-e^2}$, then the equation for the ellipse can be written

6-5c $$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.}$$

Then $c = ae$ and we may rewrite $F(ae,0)$ and the directrix DD' as $x = \frac{c}{e^2} = \frac{ae}{e^2} = \frac{a}{e}$.

Since $e < 1$, $\frac{a}{e} > a$ and the graph is shown in Fig. 6-5c.

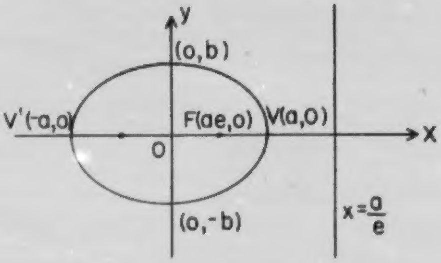


Fig. 6-5c

The line segment V'V with length 2a is called the major axis. It is a line of symmetry of the curve and passes through the focus and is perpendicular to the directrix. The line joining the points (0,-b) and (0,b) having length 2b is called the minor axis and is parallel to the directrix. The two axes intersect in a point (0,0) which is called the center of the ellipse. We have already noticed that the graph of the equation 6-5c is symmetrical with respect to both coordinate axes and the origin. The major axis and the minor axis are axes of symmetry for the curve.

Example 6-5a. Find the coordinates of the vertices, the focus, the eccentricity, and the equation of the directrix for the ellipse whose equation

$$\frac{x^2}{25} + \frac{y^2}{9} = 1$$

Solution: $a = 5$, $b = 3$.

$b = a\sqrt{1 - e^2}$ or

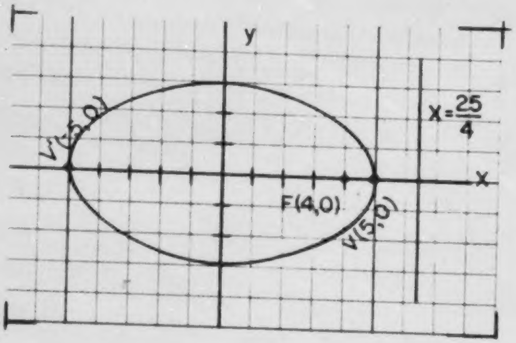
$b^2 = a^2 - a^2e^2$, and since

$c = ae$

$b^2 = a^2 - c^2$ or

$a^2 = b^2 + c^2$.

Note that the semi-major axis a is always greater than b.



Then $25 = 9 + c^2$ or $c = 4$.

Since $c = ae$ $\qquad 4 = 5e$ or $e = \frac{4}{5}$

Then the vertices are (5,0) and (-5,0); the focus is (4,0); $e = \frac{4}{5}$; the directrix $x = \frac{25}{4}$.

If we had used the point $F'(-ae,0)$ as the focus and the line $x = -\frac{a}{e}$ as the directrix, we would have obtained exactly the same ellipse. The fact leads us to state an interesting property of the ellipse: *The sum of the distances from $F(ae,0)$ and $F'(-ae,0)$ to any point on the ellipse is constant and equal to $2a$, the length of the major axis.* (The proof is left as an exercise. See Problems 8 and 9, Exercise 6-5.)

This property suggests an easy mechanical way to construct an ellipse. Take a string of length $2a$ with a loop on each end. Fasten the loops at points $(ae,0)$ and $(-ae,0)$ with thumb tacks. Place a pencil inside the string and trace the curve, keeping the string taut. The resulting curve will be the desired ellipse. There are obvious applications of this technique to constructing elliptical flower beds, patios, etc.

Now that we have more information about the ellipse, we are in a better position to discuss *the relation of the circle to the ellipse*. The shape of the ellipse depends on the constant e. If e is very close to zero, $b = a\sqrt{1 - e^2}$ is very close to a. In fact if we let e approach 0 the ellipse becomes more and more like a circle; so that we say the circle is a limiting form of an ellipse. (If e approaches 0 then c and $-c$ both approach 0 and the two foci converge at the center. The directrices $x = \pm \frac{a}{e}$ on the other hand, recede farther and farther from the foci.) This, then is the way in which the circle is related to the ellipse.

Another interesting physical property of an ellipse is the fact that a ray of light or a sound wave emanating from one focus F is reflected back from an elliptical surface to the other focus, F'. This property is responsible for the so-called whispering gallery properties of some elliptical shaped domes. A whisper at one focus can be heard distinctly by a person standing at the other focus, although the distance between the two persons may be very great.

Exercises 6-5

1. Find an equation of a circle having these properties. Sketch the graph of each on the same set of coordinate axes.

	Radius	Center
(a)	3	(0,0)
(b)	3	(0,2)
(c)	3	(2,0)
(d)	3	(3,-1)
(e)	3	(-1,2)

2. From the following equations, find the center and radius of each circle:

(a) $x^2 + y^2 = 25$

(b) $(x - 2)^2 + (y + 3)^2 = 9$

(c) $(x + \frac{1}{2})^2 + (y - 6)^2 = 5$

(d) $3x^2 + 3(y + 5)^2 = 7$

(e) $9(x + 4)^2 + 9y^2 = 36$

(f) $x^2 - 10x + 25 + y^2 = 7$

(g) $x^2 - 6x + 9 + y^2 - 8y + 16 = 16$

(h) $x^2 + y^2 - 2x + 4y + 5 = 7$

(i) $x^2 + y^2 - 4x + 6y - 1 = 0$

(j) $3x^2 + 3y^2 - 6x - 36y + 36 = 0$

(k) $x^2 - x + y^2 - 3 = -7y$

(l) $4x^2 + 4y^2 + 12x = 16y + 11$

3. Find the coordinates of the vertices, of the focus, the eccentricity, the length of the major and minor axes and the equation of the directrix for the ellipse whose equation is,

(a) $\frac{x^2}{16} + \frac{y^2}{9} = 1$

(b) $\frac{x^2}{4} + \frac{y^2}{16} = 1$

(c) $9x^2 + 25y^2 = 225$

(d) $25x^2 + 4y^2 = 100$

(e) $4x^2 + 9y^2 = 36$

(f) $2x^2 = 50 - y^2$

(g) $y^2 = 36(1 - x^2)$

(h) $\frac{x^2}{4} + \frac{y^2}{3} = 2$

(i) $\frac{x^2}{6} = 2 - y^2$

(j) $\frac{y^2}{6} = 2 - x^2$

(k) $\frac{2x^2}{3} + \frac{3y^2}{4} = 1$

4. Find the coordinates of the vertices, the focus, and the equations of the directrices of the ellipses having given the following. Sketch the graph and write an equation for each if $a = 5$ and,

(a) $e = .2$

(b) $e = .4$

(c) $e = .6$

(d) $e = .8$

5. Find an equation of the ellipse given the following:

(a) One focus $(2,0)$ and vertices $(\pm 5,0)$.

(b) Coordinates of the end points of the minor axis $(0,\pm 2)$ and of the major axis $(\pm 4,0)$.

(c) Vertices $(\pm 7,0)$ and eccentricity equal to $\frac{2}{3}$.

(d) Coordinates of the endpoints of the minor axis $(0,\pm\sqrt{3})$ and eccentricity equal to $\frac{1}{2}$.

(e) Focus $(6,0)$ and eccentricity equal to $\frac{3}{4}$.

(f) Focus $(8,0)$ and directrix $x = 10$.

(g) Vertices $(\pm 3,0)$ and directrix $x = 6$.

6. The foci of the ellipse whose equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are F and F'. What change occurs in this conic,

   (a) As $d(F,F')$ approaches 0?

   (b) As $d(F,F')$ approaches $2a$?

7. Show that for any ellipse having center at the origin the distance from either end of the minor axis to either foci is one half the major axis.

8. Given the ellipse $\frac{x^2}{16} + \frac{y^2}{12} = 1$. Show that for any point $P(x,y)$ on the ellipse, the sum of the distances from $F(2,0)$ and $F'(-2,0)$ is 8.

*9. Given the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Show that for any point $P(x,y)$ on the ellipse, the sum of the distances from $F(c,0)$ and $F'(-c,0)$ is $2a$.

10. In this Section 6-5, the focus was taken on the positive axis at $F(ae,0)$ and the directrix was always the line $x = \frac{a}{e}$. The curve would remain the same if the names of the axes should be interchanged. For example, suppose the focus is the point $F(0,1)$, the directrix is the line whose equation is $y = 4$, and the eccentricity $e = \frac{1}{2}$. Find the equation of the ellipse.

11. Compare the equation for the ellipse in problem 9 with the one for the ellipse with focus $F(1,0)$, the directrix whose equation is $x = 4$, and the eccentricity $e = \frac{1}{2}$.

12. Find an equation of the ellipse with focus $F(0,ae)$, directrix $y = \frac{a}{e}$, and eccentricity e. (Note: The major axis is still the axis containing the focus, it is perpendicular to the directrix, and always has length $2a$.)

*13. An ellipse with eccentricity e, coordinates of the center (h,k) and of the focus $F(h + c, k)$, and the equation of the directrix $x = h + \frac{c}{e^2}$. If $c = ae$ and $b = a\sqrt{1 - e^2}$, show that the equation of the ellipse can be written in the form,

$$\frac{(x - h)^2}{a^2} + \frac{(y - k)^2}{b^2} = 1$$

Note: The center of an ellipse is the midpoint of both the major and minor axis.

14. Write an equation of the ellipse from each of the following data: (Use the result of Problem 13.)

(a) Vertices $(5,2)$ and $(-3,2)$, one focus at $(4,2)$

(b) Foci $(4,3)$ and $(4,-1)$, eccentricity equal to $\frac{1}{3}$.

(c) Vertices $(-5,3)$ and $(-5,1)$, eccentricity equal to $\frac{2}{3}$.

(d) Major axis equal to 10 and parallel to y-axis, minor axis equal to 6, center $(-2,-1)$.

(e) Endpoints of minor axis at $(-3,5)$ and $(-3,-6)$, one focus at $(3, -\frac{1}{2})$.

(f) Endpoints of major axis at $(2,-3)$ and $(-12,-3)$, eccentricity equal to $\frac{5}{7}$.

*(g) Vertices $(\pm 3,2)$, directrix $x = 7$.

*(h) Focus $(3,4)$, directrix $y = 5$. (Is there more than one solution?)

*(i) Focus (-5,2), eccentricity equal to $\frac{1}{5}$. (Is there more than one solution?)

15. For each of these ellipses give the coordinates of the vertices and of the focus, the eccentricity, and an equation of the directrix. Sketch each curve showing the vertices, the focus, and the directrix. (Use the results of problem 12.)

(a) $\frac{(x - 3)^2}{25} + \frac{(y - 5)^2}{9} = 1$

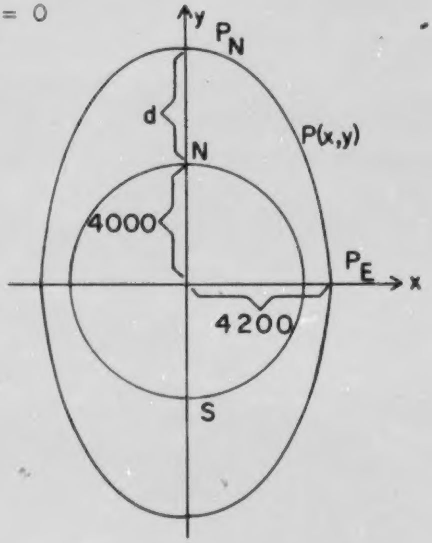
(b) $\frac{(x + 2)^2}{9} + \frac{(y - 1)^2}{16} = 1$

(c) $x^2 + 4y^2 + 6x + 9 = 16$

(d) $16x^2 + 9y^2 - 96x + 72y + 144 = 0$

(e) $4x^2 + 9y^2 + 8x - 36y + 4 = 0$

16. An artificial satellite is placed in an elliptical orbit about the earth so that the North and South poles of the earth lie in the plane of its orbit. Its distance from the North Pole plus its distance from the South Pole is constant. How high will it be when it passes directly over the North Pole, if it is 200 miles above the surface of the earth the moment when it passes through the plane of the equator? Write an equation for its orbit with respect to the center of the earth. (Assume that the diameter of the earth is 8,000 miles and that the earth is spherical.)



17. Arcs in the form of a semi-ellipse were noticed in a building. When measured, the distance across the base of the arc was found to be 24 feet and the maximum height from the base was found to be 8 feet. Find the height of the arc at intervals of 4 feet from one end to the middle.

*18. Find the coordinates of four points on the curve of $x^2 + 4y^2 = 80$ so that they are the vertices of a square having diagonals through the origin.

6-6. The Hyperbola.

In section 6-4 we defined the hyperbola as the conic with eccentricity $e > 1$; that is, the set of all points P with the property that the distance from P to a fixed point, the focus, is a constant, $e > 1$, times the distance from P to a fixed line, the directrix.

We recall that if we let the focus be $F(c,0)$ and the directrix be the line whose equation is $x = \frac{c}{e^2}$, then the x-intercepts were $\pm a = \pm \frac{c}{e}$, and although there were no y-intercepts, we let $b = \frac{c}{e}\sqrt{e^2 - 1}$. The equation of the hyperbola then assumed the simple form

6-6a $$\boxed{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.}$$

Just as for the ellipse, $c = ae$, and the focus becomes the point $F(ae,0)$ and the directrix the line $x = \frac{a}{e}$. In contrast to the ellipse, we now have $e > 1$ and $\frac{a}{e} < a$. Whereas the directrix was to the right of the focus for the ellipse, their positions are just reversed for the hyperbola. The graph is shown in Fig. 6-6a.

The line segment V'V is called the transverse axis and has length 2a. (The line segment joining the points (0,-b) and (0,b) is sometimes called the conjugate axis and has length 2b.) The origin O is called the center of the hyperbola. Again, as was the case for the ellipse, the curve is symmetric with respect to both the coordinate axes and the origin. As before we might have taken the focus to be the point F'(-ae,0) and the directrix to be the line $x = -\frac{a}{e}$ and we would have obtained the same curve.

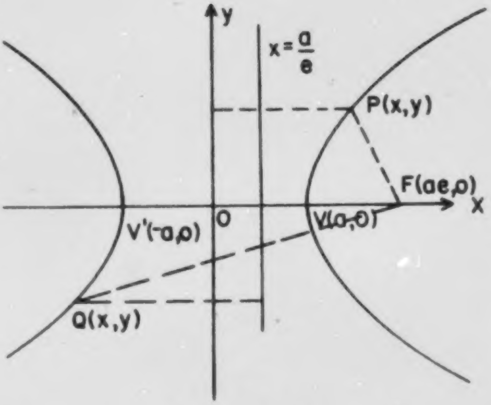
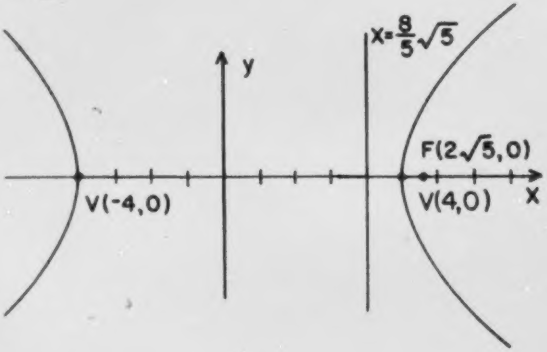


Fig. 6-6a

Example 6-6a. What are the coordinates of the vertices and the focus, the equation of the directrix, and the eccentricity for the hyperbola $\frac{x^2}{16} - \frac{y^2}{4} = 1$.



Solution:

$b^2 = a^2(e^2 - 1)$

$b^2 = a^2e^2 - a^2$

Since $c = ae$

$b^2 = c^2 - a^2$

$c^2 = a^2 + b^2$

In this problem, $a = 4$, $b = 2$

Hence, $c^2 = 16 + 4 = 20$

$c = 2\sqrt{5} = ae = 4e$

$\therefore e = \frac{1}{2}\sqrt{5}$

The vertices are $(-4,0)$ and $(4,0)$.
The focus is $(2\sqrt{5},0)$.
The equation of the directrix is $x = \frac{4}{\frac{1}{2}\sqrt{5}} = \frac{8}{5}\sqrt{5}$.

The hyperbola has a property similar to the property we noted for the ellipse, namely, the absolute value of the difference between the distances from F and F' to any point on the hyperbola is constant, and equal to $2a$. See Problems 4 and 5 of Exercise 6-6. This property is the basis for the LORAN system of navigation used extensively in World War II.

We have noticed that there are no y-intercepts for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If we solve for y we get $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$. Now if $|x| < a$, $\sqrt{x^2 - a^2}$ is not real. This shows that there is a vertical strip from $x = -a$ to $x = a$ in which there are no points on the graph of the hyperbola. On the other hand if we take larger and larger values for x, y also increases in absolute value. While these facts are extremely useful in sketching the graph, there is still another property of the hyperbola which is even more helpful for this purpose.

Example 6-6b. Sketch the graph of $\frac{x^2}{1} - \frac{y^2}{4} = 1$.

Solution: Since the curve is symmetric with respect to both coordinate axes and the origin, we need only consider the part of the graph in the first quadrant. The x-intercepts are 1 and -1. There are no points on the graph in the strip between the vertical lines $x = -1$ and $x = 1$. See Fig. 6-6b.

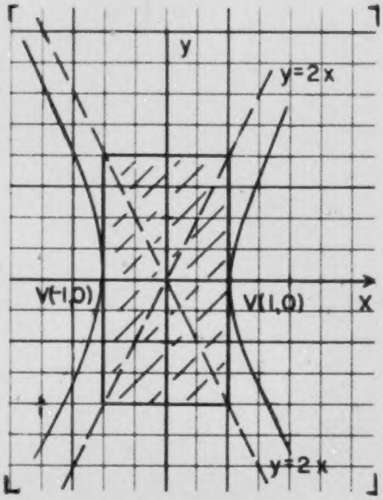


Fig. 6-6b

Solving for y we get

$$y = \pm 2\sqrt{x^2 - 1}.$$

For very large values of x, y in the first quadrant is very nearly equal to $2x$. Similarly in the fourth quadrant for large x, y is close to $-2x$. We notice that the lines whose equations are $y = 2x$ and $y = -2x$ are the diagonals of the rectangle with sides of length $2a = 2$ and $2b = 4$, parallel to the coordinate axes and centered at the origin. These two lines are called asymptotes of the hyperbola. We use the fact that the curve gets closer and closer to these lines as x increases, to sketch the graph in the first and fourth quadrants. The rest of the curve can be drawn using the symmetry of the curve.

Let us turn now to the equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If we solve for y we get $y = \pm \frac{b}{a}\sqrt{x^2 - a^2}$. In the same way, since a is a constant, if we take large values for x, then y in the first quadrant is nearly equal to $\frac{b}{a}x$. In the fourth quadrant y is close to $-\frac{b}{a}x$. The lines whose equations are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ are called asymptotes of the hyperbola.

We notice as before that these lines are diagonals of the rectangle with sides of length $2a$ and $2b$ parallel to the axes, centered at the origin. These equations can be written

$$0 = \frac{b}{a}x - y \quad \text{and} \quad \frac{b}{a}x + y = 0.$$

Both lines are the graph of the single equation

$$\frac{b^2}{a^2}x^2 - y^2 = 0$$

$$\text{or} \qquad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0.$$

The fact that the hyperbola has these lines as asymptotes greatly reduces the work involved in sketching its graph. We simply plot the points which are the vertices and use the asymptotes (that is, the diagonals of the rectangle) to sketch the curve. As a rule no other points need to be plotted.

Example 6-6c. Sketch the graph $\frac{x^2}{25} - \frac{y^2}{9} = 1$.

Solution: The vertices are $(-5,0)$ and $(5,0)$. The asymptotes are $y = \pm\frac{3}{5}x$. See Fig. 6-6c.

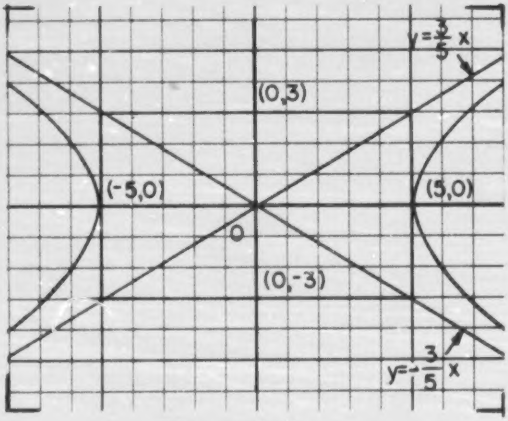


Fig. 6-6c

Although we said in Section 6-4 that every equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ has as its graph a conic section (or a limiting form of one of these), we have not encountered any equation in which B was not zero. This is because we have always considered conics with axes parallel to the coordinate axes. If we had taken more general positions for the directrix, B would not have been zero. In particular if the transverse axis of the hyperbola is the line $y = x$ and the center is at the origin, the equation of the hyperbola may take the form $xy = k$.

A hyperbola with an equation of this form is called an equilateral or rectangular hyperbola. Fig. 6-6d shows the graph of this equation for $k = 1,\ 2,\ 3$.

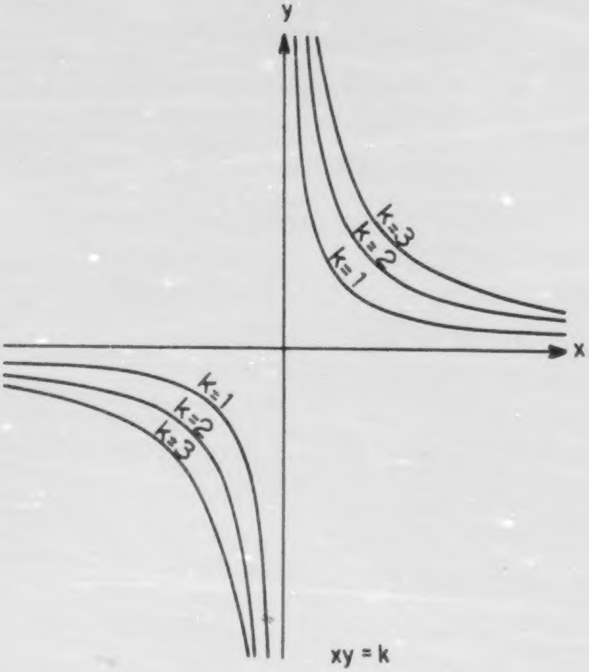


Fig. 6-6d

Example 6-6d. There is a scientific principle, important in both physics and chemistry, known as Boyle's law, which may be stated as follows: If a fixed mass of gas is confined in a cylinder with a piston (Fig. 6-6e), and if a variable pressure p is applied to the piston, resulting in a corresponding change in the volume v, then p and v are related by the equation $pv = k$, where the particular value of the constant k will depend on the kind of gas as well as on other factors. If we let the positive x-axis be the p-axis and the positive y-axis be the v-axis, then the equation $pv = k$ will be represented by one branch of an equilateral hyperbola.

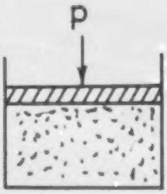


Fig. 6-6e

When two variables x and y are related by an equation of the form $xy = k$, or $y = \frac{k}{x}$, then y is said to vary inversely as x, or to be inversely proportional to x. Thus, for example, Boyle's law asserts that v is inversely proportional to p.

Example 6-6e. Suppose that a certain mass of gas is confined, at pressure of 10 pounds per square inch, in a volume of 200 cubic inches. Find the relation between pressure and volume, and determine the volume when the pressure is increased to 50 pounds per square inch.

Solution: The constant k in the equation $pv = k$ is determined by substituting the values $p = 10$ and $v = 200$; thus $k = 2000$. The relation between p and v may be written as $v = 2000/p$, so that when $p = 50$, $v = 40$ cubic inches.

Exercises 6-6

1. Given $\frac{x^2}{9} - \frac{y^2}{4} = 1$.
   (a) Write an equation of each asymptote of the hyperbola.
   (b) Give the coordinates of the vertices.
   (c) Sketch the graph of the equation.
2. Given $3xy = 36$.
   (a) Write an equation of each asymptote.
   (b) Give the coordinates of the vertices.
   (c) Sketch the graph of the equation.

3. Give the coordinates of the vertices, the coordinates of the focus, an equation of the directrix, an equation of the asymptotes and the eccentricity of the following. Sketch the curve showing the vertex, the focus, the directrix, and the asymptotes. [See examples in 6-6.]

(a) $\frac{x^2}{16} - \frac{y^2}{25} = 1$

(b) $\frac{x^2}{25} - \frac{y^2}{16} = 1$

(c) $x^2 - y^2 = 36$

(d) $y^2 - x^2 = 36$

(e) $4y^2 - 3x^2 = 36$

(f) $3x^2 - 4y^2 = 36$

4. Given the hyperbola $\frac{x^2}{1} - \frac{y^2}{4} = 1$. Show that the absolute value of the difference of the distances from any point in the first quadrant on the hyperbola to the points $F(\sqrt{5},0)$ and $F'(-\sqrt{5},0)$ is 2.

5. Given the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Show that the absolute value of the difference of the distances from any point P on the hyperbola to the points $F(ae,0)$ and $F'(-ae,0)$ is $2a$. (Hint: $b^2 = a^2(e^2 - 1)$ or $a^2 + b^2 = a^2e^2$. Since $c = ae$, $a^2 + b^2 = c^2$).

6. Write in standard form the equation of the hyperbola given the coordinates of the foci and the absolute value of the difference of the distances from point $P(x,y)$ on the hyperbola to the two foci:

(a) $F(4,0)$, $F'(-4,0)$, and $|d(P,F)-d(P,F')| = 6$.

(b) $F(0,4)$, $F'(0,-4)$, and $|d(P,F)-d(P,F')| = 6$.

7. Suppose we take the focus of the hyperbola to be the point $F(0,2)$, the directrix to be the line whose equation is $y = \frac{1}{2}$, and the eccentricity to be 2. Find the equation of the hyperbola and sketch its graph.

8. Derive an equation of the hyperbola with focus $F(0,ae)$, directrix $y = \frac{a}{e}$, and eccentricity e. (Note that the vertices are $(0,a)$ and $(0,-a)$. The transverse axis is on the y-axis and has length $2a$. Hint for the solution: Let $b^2 = a^2(e^2 - 1)$.)

9. Write an equation of the hyperbola from the given set of data.
   (a) Vertices $(\pm 5,0)$ foci $(\pm 8,0)$.
   (b) Vertices $(\pm 3,0)$ distance between foci equal to 8.
   (c) Vertices $(\pm 3,0)$ eccentricity equal to 2.
   (d) Directrices $x = \pm 2$, one vertex at $(4,0)$.
   (e) Foci $(\pm 7,0)$ eccentricity equal $\frac{4}{3}$.
   (f) Asymptotes $y = \pm 3x$, one vertex at $(2,0)$.
   (g) Asymptotes $3x + 2y = 0$ and $3x - 2y = 0$, focus $(0,3)$.

10. Sketch the graph for each of the following, making use of the asymptotes, vertices, and when necessary a few sample points.
   (a) $9x^2 - 4y^2 = 36$
   (b) $4x^2 - y^2 = 4$
   (c) $4x^2 - 9y^2 = 36$
   (d) $x^2 - 4y^2 = 4$
   (e) $9x^2 - y^2 = 9$
   (f) $xy = 4$
   (g) $y^2 - 9x^2 = 9$
   (h) $xy - 1 = 0$
   (i) $xy + 4 = 0$
   (j) $25x^2 - 4y^2 - 100x + 40y = 100$

11. Sketch the graph of,
   (a) $y = \sqrt{36 + x^2}$
   (b) $y = -\sqrt{36 + x^2}$
   (c) $x = \sqrt{36 + y^2}$
   (d) $x = -\sqrt{36 + y^2}$

   Are any of these hyperbolas? Explain.

12. Find an equation of a hyperbola passing through the point of $(2,3)$ having as asymptotes the lines of $3x - 5y = 0$ and $3x + 5y = 0$.

13. Find an equation of the hyperbola whose asymptotes are the lines of $9x^2 - 25y^2 = 0$ and which passes through the point of $(5,1)$.

14. Find an equation of the hyperbola through the point of $(0,2)$ and having as asymptotes the lines of $4x - y = 0$ and $4x + y = 0$. Sketch the curve.

15. If a hyperbola passes through the point of $(2,0)$ and has asymptotes of $y = \pm 4x$ find an equation of the hyperbola and sketch the curve.

*16. Show that the hyperbola with center at $A(h,k)$ and focus at $F(h + ae,k)$, directrix $x = h + \frac{a}{e}$, and eccentricity e has the equation,

$$\frac{(x - h)^2}{a^2} - \frac{(y - k)^2}{b^2} = 1.$$

(Note: The center of a hyperbola is the point of intersection of its asymptotes.)

17. Write an equation of the conic for each set of data:

(a) Vertices $(3,1)$ and $(0,1)$, one focus at $(4,1)$.

(b) Foci $(4,-3)$ and $(-2,-3)$, one vertex at $(3,-3)$.

(c) Foci $(5,7)$ and $(-2,7)$, eccentricity equal $\frac{5}{4}$.

(d) Vertices $(-5,7)$ and $(2,7)$, eccentricity equal $\frac{3}{2}$.

18. Find an equation of the hyperbola with center at $C(3,4)$, focus $F(3 + 2\sqrt{2},4)$, and directrix whose equation is $x = 3 + \sqrt{2}$. Sketch the graph.

19. Find the coordinates of the vertices, the focus, the center; an equation of the directrix, the asymptotes; and the eccentricity of the following:

    (a) $\frac{(x + 2)^2}{9} - \frac{(y - 5)^2}{4} = 1$

    (b) $\frac{(y - 4)^2}{16} - \frac{(x - 1)^2}{1} = 1$

    (c) $\frac{y^2}{16} - \frac{x^2}{25} = 1$

    (d) $3(x - 3)^2 - 2(y + 2)^2 = 18$

    (e) $x^2 - y^2 - 4x + 6y - 6 = 0$

    (f) $x^2 - 2x - y^2 - 6y - 17 = 0$

    (g) $9x^2 - 72x - 16y^2 - 96y = 144$

    (h) $4y^2 + 12y + 12 - x^2 + 4x + 9 = 0$

    (i) $9y^2 - 4x^2 - 4x - 18y + 44 = 0$

    (j) $4x^2 - 25y^2 + 32x = -50y - 39$

20. If y varies inversely as x, and $y = 4$ when $x = 2$,

    (a) Find the relation between x and y.

    (b) Find the value of y when $x = 5$.

    (c) Draw a graph for (a) and use it to check your answer for (b).

21. For a given electromotive force the current I carried by a wire varies inversely as the resistance R. With a certain electromotive force a wire whose resistance is 15 ohms will carry a current of 20 amperes. Find the current produced by the same electromotive force if the resistance is increased to 50 ohms.

22. If the relation between variables x and y is of the form $y = \frac{k}{x^2}$, then y is said to vary inversely as the square of x,

    (a) Find the value of k if $y = 3$ when $x = 5$.

    (b) Find the value of y when $x = 2$.

23. According to Newton's law of gravitation, the weight of a body varies inversely as the square of its distance from the center of the earth. If a body weighs 50 pounds at the surface of the earth, how much would it weigh at a height of 200 miles above the earth, assuming the radius of the earth to be 4000 miles.

*24. On a level plane the sound of a rifle and that of its bullet striking the target are heard at the same instant. Describe the possible set of locations of the listener.

6-7 Supplementary Exercises

1. Find the slope and the intercepts of each line having the following equation:

(a)	$3y - 2x - 15 = 0$	(e)	$\frac{x}{6} - \frac{y}{2} = 1$
(b)	$3x + 2y - 20 = 0$	(f)	$y = 5$
(c)	$2y - 3x - 8 = 0$	(g)	$\frac{2x}{5} + \frac{y}{3} = 1$
(d)	$\frac{2x}{3} = 0$	(h)	$\frac{3}{4}y - \frac{2}{7}x - 1 = 0$

2. Write an equation of the line having intercepts whose coordinates are $(6,0)$ and $(0,-\frac{5}{2})$.

3. Write an equation of the line passing through the points $A(-1,-5)$ and $B(\frac{2}{3}, -5)$.

4. Write an equation of the line having the following properties:

   (a) Slope $m = 3$, y-intercept is 2.

   (b) Parallel to line $x + y - 6 = 0$, y-intercept is 4.

   (c) Perpendicular to line $2x + 3y - 22 = 0$, y-intercept is -3.

5. Consider the equations,

   (1) $y = x + 6$

   (2) $y = 3x$

   (3) $y = \frac{2}{3}x - 4$

   (4) $y = \frac{3}{4}x$

   (5) $y = -\frac{5}{6}x + 2$

   (a) Write the slopes of the lines given by these equations.

   (b) What is the y-intercept of each line?

   (c) Which of the lines rise to the right, and which ones sink to the right?

   (d) Which of the 5 lines is the steepest?

   (e) Find an equation of the line which has the same slope as the line defined by (1) and the same y-intercet as the line defined by (3).

6. Write an equation of the line through $(-2,3)$ and parallel to the line $y = 3x - 8$.

7. Write an equation of the line perpendicular to the line $2y = 5x + 10$ and passing through the point $(+3, -2)$.

8. The coordinates of the vertices of a triangle are $A(5,10)$, $B(10,-7)$, and $C(-5,-5)$. Write an equation of the lines forming this triangle.

9. Determine without sketching the graph which pairs of equations represent lines which are parallel, perpendicular, the same, or neither.

(a) $2x - 3y = 5$

$3x + 2y - 4 = 0$

(b) $2x - 4 + 3y = 0$

$3x - 7 = -2y$

(c) $x + 2y = 6$

$2x = 6 + y$

(d) $2x = 3 + y$

$y = 2x - 5$

(e) $y = 7$

$y = 12$

(f) $2x + y - 7 = 0$

$2x - y + 7 = 0$

(g) $2x + y = 0$

$2x = 1$

(h) $x - 2y + 5 = 0$

$y = \frac{1}{2}$

10. Given,

(i) $3x - 2y - 4 = 0$ (ii) $3x + 4y + 12 = 0$

(a) Determine the y-intercept b and the slope m of each.

(b) Sketch the graph of each on the same coordinate axes.

(c) Are these lines perpendicular? Explain.

11. Write an equation of a line passing through the point $(0,0)$ and perpendicular to the line whose equation is $2x + y - 4 = 0$.

12. Write an equation of a line passing through the point $(2,-1)$ and parallel to the line whose equation is $\frac{x}{-3} + \frac{y}{2} = 1$.

13. Write an equation of a line through the point $(-5,1)$ and having the same y-intercept as the line whose equation is $2x + y - 4 = 0$.

14. Write an equation of a line parallel to the line whose equation is $3y = x$ and passes through the x-intercept of the line whose equation is $x + 3y = 3$.

15. If the speed of reaction of 2 chemicals doubles for every 10° C rise in temperature t on the range $0^\circ C \leq t \leq 100^\circ C$, how many times as fast would the reaction proceed at $100^\circ C$ than at $20^\circ C$?

16. If A varies directly as C, and B varies directly as C, show that $A + B$, $A - B$ and $\sqrt{AB}$ will each vary directly as C.

17. Write an equation of the curve having,

    (a) $F(3,0)$, $F'(-3,0)$, and $d(P,F) + d(P,F') = 10$.

    (b) $F(0,3)$, $F'(0,-3)$, and $d(P,F) + d(P,F') = 10$.

    (c) $F(3,0)$, $F'(-3,0)$, and $\left|d(P,F) - d(P,F')\right| = 2$.

    (d) $F(0,3)$, $F'(0,-3)$, and $\left|d(P,F) - d(P,F')\right| = 2$.

    (e) $F(0,1)$, $Q(x,-1)$, and $d(P,F) = d(P,Q)$.

    (f) $F(1,0)$, $Q(-1,y)$, and $d(P,F) = d(P,Q)$.

    Identify each.

18. Identify the conic whose equation is,

    (a) $9x^2 + 9y^2 = 4$.

    (b) $2x^2 + 3y^2 = 6$.

    (c) $4x^2 - 16y^2 = 16$.

    (d) $4x^2 + 16y^2 = 16$.

    (e) $4x^2 + 16y = 0$.

    (f) $y^2 = 9x - 36$.

    (g) $x^2 + y^2 + 4y = 0$.

    (h) $x^2 + 4y^2 + 6x + 9 = 0$.

    (i) $9x^2 - 16y - 72x + 96y = 144$

    (j) $x^2 - 4y^2 + 2x + 16y - 19 = 0$.

    (k) $9y^2 + 16x^2 - 96x + 72y = -144$

    (ℓ) $y^2 + 3x + 6y = 0$.

    (m) $4x^2 - 8x - 36y = 9y^2 + 68$.

19. Graph each of the following:

(a) $y^2 = (x - 5)^2$

(b) $x^2 = (y - 1)^2$

(c) $(2x - y)^2 = 4$

(d) $9 - (x - 2y)^2 = 0$

(e) $x^2 = y^2$

(f) $xy = 0$

*(g) $x^2 + 2xy + y^2 - 4 = 0$

*(h) $4x^2 + 9y^2 = 9 - 12xy$.

(i) $x < 0$

(j) $y \geq 0$

(k) $x^2 + 4y^2 \leq 4$

(ℓ) $4x^2 + y^2 \geq 4$

(m) $x^2 + y^2 \leq 16$

(n) $x^2 - 4y^2 < 4$

(o) $4y^2 - x^2 > 4$

(p) $y^2 - 4x \geq 0$

(q) $x^2 - 4y \leq 0$

*(r) $x^2 + y^2 - 4x + 6y + 13 \leq 0$.

*(s) $y^2 - 4x - 4y < 0$.

*(t) $\{(x,y) \mid x < 0\} \cup \{(x,y) \mid y \leq 0\}$

20. Find an equation of a circle which has as a diameter the latus rectum of the parabola whose equation is $y^2 = 16x$.

21. Find an equation of the hyperbola whose asymptotes are the lines of $3x^2 - 5y^2 = 0$ and which passes through the point of $(2,3)$.

22. If the asymptotes of a hyperbola are given by $2x^2 - 7y^2 = 0$ and the hyperbola passes through the point of $(3,0)$, find an equation of the hyperbola.

23. Find the equation of the hyperbola whose asymptotes are given by $a^2x^2 - b^2y^2 = 0$, and which passes through the point of $(b,0)$. (a and b are real numbers).

24. Sketch the graph of $x^2 - y^2 = k$, when k has the following values:

(a) $k = 16$	(e) $k = -1$
(b) $k = 9$	(f) $k = -4$
(c) $k = 4$	(g) $k = 0$
(d) $k = 1$	(h) $k = -16$

*25. Find the coordinates of the end points of the chords perpendicular to the transverse axis at the foci of the hyperbola whose equation is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. Develop a formula for the length of these chords in terms of a and b. Will this same formula hold for a hyperbola of $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$?

26. Find an equation of the set of points $P(x,y)$ such that the distance from P to the vertex of the parabola $x^2 = 8y$ is twice the distance to its focus.

27. The arch of a stone bridge has the form of a parabola; the span is 40 feet, and the maximum height is 10 feet. Find the height of the arch at intervals of 5 feet from one end to the middle.

28. Show that if a parabola has its vertex at (a,b) and focus at $(a+c,b)$, then the equation of the parabola is $(y-b)^2 = 4c(x-a)$.

29. Find an equation similar to that of 2 for a parabola having vertex at (a,b) and focus at $(a,b+c)$.

30. Show that if a parabola has its vertex at (a,b) and the line of $x = a - c$ as directrix, an equation of the parabola is $(y - b)^2 = 4c(x - a)$.

Challenge Problems

1. Find the equation of the parabola having $x = -a$ as directrix and focus at $(a,0)$. Discuss the curve for $a > 0$. For $a < 0$.

2. A chord through the focus perpendicular to the axis of a parabola is called the focal chord of the parabola. Show that the end points of the focal chord of the parabola $y^2 = 4ax$ are $(a,2a)$ and $(a,-2a)$.

3. Find the equation for the parabola with focus at $(1,1)$ and the line of $y = -x$ as directrix. Sketch the curve.

4. Find the equation of the line parallel to the line whose equation is $y = \frac{1}{2}x + 2$ which is 2 units from this line.

5. The line through the focus F and the point $P_1(x_1,y_1)$ on the parabola $y^2 = 4cx$ intersects the parabola in a second point $P_2(x_2,y_2)$. Find the coordinates of P_2 in terms of x_1, y_1, and c. If V is the vertex, the line through P_1V cuts the directrix at R, prove that the line through P_2R is parallel to the axis of the parabola.

Chapter 7

SYSTEMS OF EQUATIONS IN TWO VARIABLES

7-1. Solution Sets of Systems of Equations and Inequalities.

Definition 7-1a. The solution set of an equation (inequality) in two variables x and y is the set of ordered pairs of real numbers (x,y) which satisfy the equation (inequality).

The same one-to-one correspondence which we set up in Chapter 2 between ordered pairs of real numbers and the points in the plane now gives us a one-to-one correspondence between the elements of the solution set of an equation (inequality) and the points on the graph of the equation (inequality).

In this chapter we are again interested in the algebraic aspects of equations; that is, the ordered pairs of real numbers which satisfy the equations. However, we will freely use whatever geometric information we may have about an equation to determine its solution set.

Example 7-1a. Find the solution set of,

(a) $y = x$

(b) $y = 2x^2$

(c) $y \geq 2x$

(d) $x^2 + y^2 > 1$.

Solution:

(a) The solution set of $y = x$ is the set of ordered pairs (a,a) where a is any real number.

(b) The solution set of $y = 2x^2$ is the set of ordered pairs $(a,2a^2)$ where a is any real number.

(c) The solution set of $y \geq 2x$ is $\{(x,y): y \geq 2x\}$.

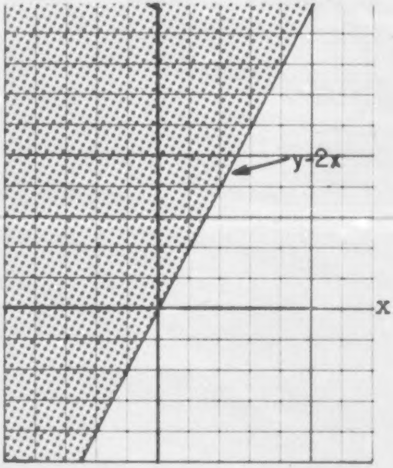


Fig. 7-1a.

However this is really just a restatement of the problem and while it is a true statement, it is not very enlightening. We use a graph to indicate the solution set. See Fig. 7-1a. We draw the graph of $y = 2x$. Then for any particular value of x, the pair of coordinates of any point (x,y) with $y \geq 2x$ corresponds to an element of the solution set. Geometrically these are the points on any vertical line $x = a$, on or above the point $(a,2a)$. Thus the graph of the inequality is the shaded region in Fig. 7-1a. The solution set is the set of ordered pairs which are coordinates of points in the shaded region.

(d) To obtain the solution set of $x^2 + y^2 > 1$, we draw the graph of $x^2 + y^2 = 1$. This is the circle with center at the origin and radius 1. If P is any point outside the circle, then

$$d(O,P) = \sqrt{x^2 + y^2} > 1.$$

Or

$$x^2 + y^2 > 1.$$

Conversely, if $x^2 + y^2 > 1$, then $\sqrt{x^2 + y^2} > 1$, $d(O,P) > 1$, and the point lies outside the circle. The solution set is the set of ordered pairs which are coordinates of points outside the circle with center at the origin and radius 1.

From now on for "the solution set is the set of ordered pairs which are the coordinates of points belonging to the set..." we shall use the less precise, but shorter "the solution set is the set of points...". This briefer statement is justified by the one-to-one correspondence which has been established between the set of ordered pairs of real numbers and the set of points in the plane.

In this chapter and the next we want to consider the set of ordered pairs which satisfy two or more equations (inequalities). When such problems are considered we shall refer to the two or more equations (inequalities) as a _system_ _of_ _equations_ (inequalities). Each of the individual equations (inequalities) is called a _component_ of the system.

Definition _7-1b_. _The solution set of a system of equations (inequalities) in two variables_ x _and_ y is the set of _all_ ordered pairs (x,y) which are common to the solution sets of the component equations (inequalities).

Suppose we are considering a system of two equations. Let the solution sets corresponding to the equations be S_1 and S_2. Then the solution set S of the system is the set of ordered pairs which are in both S_1 and S_2. (In set language, this set is called the "intersection" of S_1 and S_2. The symbol for set intersection is "$\cap$". The solution set S can then be written $S = S_1 \cap S_2$.

Example _7-1b_. What is the solution set of the following systems:

(a) $\begin{cases} x + y - 2 = 0, \\ x - y + 2 = 0. \end{cases}$ (b) $\begin{cases} |x| > 2, \\ |y| < 1. \end{cases}$

Solution:

(a) The solution set of the system is $\{(0,2)\}$; that is, the set of ordered pairs consisting of the single ordered pair $(0,2)$. We can use the graphs of the equations to convince ourselves that this is the only ordered pair of the solution set. The ordered pair $(0,2)$ is the only member of the solution set since any ordered pair in the solution set must be the coordinates of a point on both the lines $x + y - 2 = 0$ and $x - y + 2 = 0$. These lines intersect in only one point; namely $(0,2)$.

(b) The solution set of $|x| > 2$ is the set of points to the left of the line $x = -2$ and to the right of the line $x = 2$. See Fig. 7-1b. The solution set of $|y| < 1$ is the set of points inside the horizontal strip between $y = -1$ and $y = 1$. The solution set of the system is the intersection of these two sets or the set of points in the cross-hatched region in Fig. 7-1b.

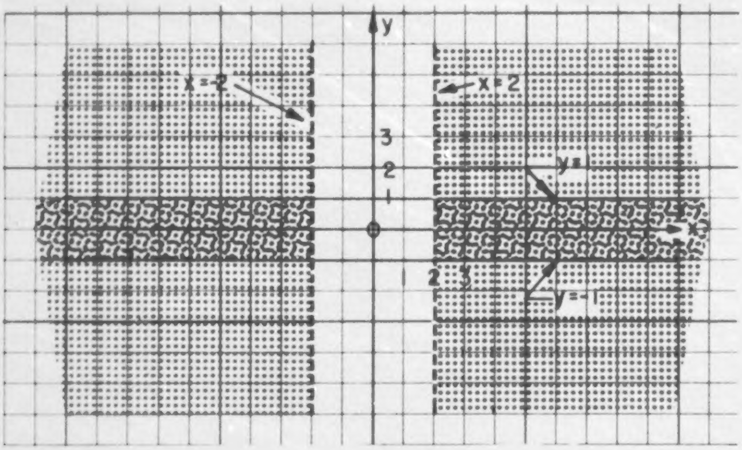


Fig. 7-1b.

Suppose we have a system consisting of two equations which have solution sets S_1 and S_2. According to our definition the solution set, S, of the system is the intersection of S_1 and S_2. If the intersection of S_1 and S_2 is the empty set, then the equations have no common solution and the system is said to be inconsistent.

The following systems are examples of inconsistant systems.

(a) $\begin{cases} 2x + y = 5 \\ 2x + y = -11 \end{cases}$ (b) $\begin{cases} x^2 + y^2 = 20 \\ x^2 + y^2 = 6 \end{cases}$

If the intersection of S_1 and S_2 is not empty, then there must be at least one pair of numbers (x,y) which will satisfy both equations. The system is then said to be consistent.

The system
$$\begin{cases} 2x - 7y = -5 \\ 5x + 3y = 8 \end{cases}$$
is consistent because we can verify the fact that the pair $x = 1$, $y = 1$ will satisfy both equations.

A consistent system is said to be dependent if $S_1 = S_2$; that is, for example, consider the system
$$\begin{cases} 3x + 7y = 12 \\ 6x + 14y = 24 \end{cases}$$
The second equation is obtained by doubling each member of the first. It is evident that any solution of the first equation is also a solution of the second and conversely. $\therefore$ the system is dependent. The graphs corresponding to the two equations are the same straight line. The system of quadratic equations
$$\begin{cases} x^2 + 4y^2 = 100 \\ 2y^2 = 50 - \frac{1}{2}x^2 \end{cases}$$
is also dependent.

We can summarize our conclusions about systems of equations in two variables as follows:

A system is inconsistent if its solution set is the empty set; that is, the component equations have no common solutions.

A system is consistent if its solution set contains at least one member; that is, the component equations have at least one common solution.

A consistent system is dependent if the solution set of the system is the same as the solution set of one of the component equations; that is every solution of any one of its component equations is a solution of every other.

Exercises 7-1

1. Is $(2,0)$ an element of the solution set of the system

$$\begin{cases} 2x + 3y = 4 \\ 8x - 7y = 16 \text{ ?} \end{cases}$$

Sketch the graph of each of the two equations. How do the graphs illustrate your answer?

2. Is $(1,2)$ an element of the solution set of the system

$$\begin{cases} 4x - y = 2 \\ 12x - 3y = 6 \text{ ?} \end{cases}$$

Sketch the graph of each of the two equations. Are there other elements which belong to the solution set of this system?

3. Does the solution set of

$$\begin{cases} x + 4y = 13 \\ 2x + 8y = 14 \end{cases}$$

contain the element $(1,3)$? Sketch the graph of each of the two equations. According to the graphs, what seems to be the solution set for the system?

4. For the system

$$\begin{cases} 3x - 4y = 11 \\ 12x - 16y = b: \end{cases}$$

(a) For what value of b will the solution set be empty?

(b) For what value of b will the solution set contain the element $(5,1)$?

(c) For what value of b will the solution set contain infinitely many ordered pairs?

5. What different types of solution sets can there be for the system

$$\begin{cases} ax + by = c \\ dx + ey = f \text{ ?} \end{cases}$$

Discuss the graphical interpretation of your answer.

6. By inspection determine which of the following systems are consistent. If the system is consistent, determine whether or not it is also dependent.

(a) $\begin{cases} x + y = 1 \\ \frac{1}{2}x = 2 - \frac{1}{2}y \end{cases}$

(b) $\begin{cases} x = 2y - 1 \\ 2x = 2y - 1 \end{cases}$

(c) $\begin{cases} y = 2x - 1 \\ x - \frac{1}{2}y = \frac{1}{2} \end{cases}$

(d) $\begin{cases} y = 2x + 1 \\ y = 2x + 3 \end{cases}$

(e) $\begin{cases} 7x + 5y = 11 \\ 3x - 2y = 13 \end{cases}$

(f) $\begin{cases} 4x = 26 + 7y \\ 5x - 7 = 11y \end{cases}$

(g) $\begin{cases} x^2 + y^2 = 34 \\ x^2 - y^2 = 16 \end{cases}$

(h) $\begin{cases} x^2 + 4y^2 = 61 \\ 9x^2 - 25y^2 = 0 \end{cases}$

(i) $\begin{cases} x^2 + y^2 = 4 \\ 2x^2 + y^2 = 4 \end{cases}$

(j) $\begin{cases} y = x^2 \\ y = x^2 + 5 \end{cases}$

(k) $\begin{cases} 3x^2 + 3y^2 = 15 \\ 2x^2 + 2y^2 = 10 \end{cases}$

(ℓ) $\begin{cases} 6x^2 - 2y + 2 = 0 \\ 9x^2 - 3y + 3 = 0 \end{cases}$

7. Does the solution set of the system

$$\begin{cases} y = x^2 \\ y = 2 - x^2 \end{cases}$$

contain the element $(1,1)$? Can you use symmetry to find a second element of the solution set? Find one. Sketch graphs of the two equations. How many solutions does the system seem to have?

8. Find the solution set by sketching the graph of each of the following:

(a) $x^2 + 4y^2 = 4$

(b) $x^2 + 4y^2 > 4$

(c) $x^2 + 4y^2 < 4$

(d) $x^2 = y^2$

(e) $x^2 \geq y^2$

(f) $xy = 0$

(g) $y^2 = (x - 5)^2$

(h) $x < 0$

(i) $y \geq 0$

(j) $x^2 = (y - 1)^2$

*(k) $x^2 + 2xy + y^2 - 4 = 0$

*(ℓ) $y < 2x^2 + 4x + 4$

*(m) $\{(x,y): x \leq 0\} \cup \{(x,y): y \leq 0\}$

(Note: $\cup$ is the symbol for union. The solution set of such a sentence consists of the elements which belong to either set.)

9. Is $(3,2)$ an element of the solution set of the system

$$\begin{cases} 2x - 3y = 0 \\ x + y - 5 = 0 \\ 5x - 3y - 9 = 0 \end{cases}$$

Sketch the graphs of these equations. How do the graphs illustrate your answer?

10. Is $(1,2)$ an element of the solution set of the system

$$\begin{cases} 3x + 2y = 5 \\ 5x - y = 3 \\ 16x + 2y = 20\ ? \end{cases}$$

Sketch the graphs of the three equations. How do the graphs illustrate your answer?

11. How must the graphs of the component equations of the system

$$\begin{cases} ax + by = c \\ dx + ey = f \\ gx + hy = k \end{cases}$$

be related if there is to be a single element in the solution set S?

12. If the system

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases}$$

has a single element in its solution set, what would you suspect about $m(ax + by - c) + n(dx + ey - f) = 0$? Test your conjecture by referring to Problem 9. Obviously, this does not constitute a proof, but can you prove it?

By our definition, the solution set of an equation is a set of ordered pairs of real numbers. Of course in the definition we might have substituted for "real numbers", elements from any number system. In particular if we allowed ordered pairs of complex numbers, some systems above which were inconsistent might have solutions. In the next two problems, use the definition: The solution set of an equation is the set of all ordered pairs of complex numbers which satisfy the equation.

*13. What is the solution set of the system:

$$\begin{cases} y = x^2 \\ y = x - 4 \quad ? \end{cases}$$

*14. What is the solution set of the system:

$$\begin{cases} ix + (2 - i)y + 6i = 0 \\ \qquad\qquad x - iy = 0 \quad ? \end{cases}$$

7-2. Equivalent Equations and Equivalent Systems of Equations.

Definition 7-2a. *Two equations (inequalities) are equivalent* if and only if they have the same solution set.

We have already been led to consider equivalent equations several times in this course. The process of solving $3x + 2 = 0$ consists of replacing the equation by the equivalent equation $x = -\frac{2}{3}$. In Chapter 6 we developed several equivalent equations for non-vertical straight lines. For example, $2x + 3y - 6 = 0$ is equivalent to $\frac{x}{3} + \frac{y}{2} = 1$ and also to $y = -\frac{2}{3}x + 2$. Each of these equivalent equations for the same straight line makes it easy for us to obtain some specific information about the line. Just as we find it useful to consider several equivalent equations for the same straight line, we shall now find it helpful to consider systems of equations which are equivalent to a given system. In fact the general method of solving systems of equations which we shall develop consists of finding particular systems which are equivalent to the given system.

First we shall define equivalent systems of equations and then we shall show how the idea of equivalent systems helps us to find the solution set of the system.

Definition 7-2b. Two systems of equations are equivalent if and only if they have the same solution set.

Example 7-2a. The system

$$\begin{cases} 3x - y - 8 = 0 \\ x + 2y - 5 = 0 \end{cases}$$

is equivalent to the simpler system

$$\begin{cases} x = 3 \\ y = 1 \end{cases}$$

which allows us to write $(3,1)$ as the solution set of the original system.

In the next several sections we shall be concerned with methods for obtaining the solution set of a given system of equations.

Before we proceed to study these methods, let us review some of the operations which lead to equivalent equations, as well as some of the operations which may not lead to equivalent equations. The following examples illustrate such operations.

Example 7-2b.

(a) $x - 2 = 0$ is equivalent to $x = 2$.

(b) $x^2 + y = 0$ is equivalent to $y = -x^2$.

(c) $\frac{1}{2}x = 6$ is equivalent to $x = 12$.

(d) $2x = 6$ is equivalent to $x = 3$.

(e) $x^2 - 2 = 0$ is equivalent to $x^2 + 2x + 1 = 2x + 3$.

Example 7-2c.

(a) $x^2 + y^2 = 0$ is not equivalent to $x(x^2 + y^2) = 0$.

Since the solution set of the first equation is $\{(0,0)\}$; while that of the second is $\{(0,y)\}$, y any real number.

(b) $x = -2y$ is not equivalent to $x^2 = 4y^2$ since the solution set of the first is $\{(a, -\frac{1}{2}a)\}$, for any real number a; while the solution set of the second, in addition to the ordered pairs in the solution set of the first equation, contains all ordered pairs $(a, \frac{1}{2}a)$, for any real number a.

(c) $x^2 - y^2 = 0$ is not equivalent to $x + y = 0$. Why?

(d) $x^2 = y^2$ is not equivalent to $x = y$, since $\{(a, -a)\}$ as well as $\{(a, a)\}$, for all real a, satisfy the first equation, but not the second.

To summarize, if we add or substract the same expression from both members of an equation, or multiply or divide both members by a non-zero constant, the resulting equation is equivalent to the original one.

On the other hand, if we square or extract the square root of both members, or multiply or divide both members of an equation by an expression involving a variable, the resulting equation may not be (and probably is not) equivalent to the original one.

We now formulate a principle which is helpful in obtaining systems of equations which are equivalent to a given system and from which it is easy to find the solution set of the original system (and incidentally of all the equivalent systems).

Principle 7-2a. If either of the equations of a system is replaced by an equivalent equation, the resulting system is equivalent to the original system.

The same is true if several equations are replaced by equivalent equations. Therefore, all the algebraic operations which produce equivalent equations will be useful to us in our efforts to find the solution set of a system.

Example 7-2d. Find the solution set of the system

$$\begin{cases} 3x + 2y + 4 = 0 & (1) \\ 5x - 3y - 25 = 0 & (2) \end{cases}$$

Solution: To eliminate y, we multiply the first equation by 3 and the second by 2 and add, obtaining the equation

(3) $3(3x + 2y + 4) + 2(5x - 3y - 25) = 0.$

Now it is clear that any pair of numbers (x_1,y_1) which satisfies the first and third equations must satisfy the second. The proof is simple. Since the pair (x_1,y_1) satisfies the first and third equations we have

$$3x_1 + 2y_1 + 4 = 0$$

and

$$3(3x_1 + 2y_1 + 4) + 2(5x_1 - 3y_1 - 25) = 0,$$

from which it follows that $5x_1 - 3y_1 - 25 = 0$. This equation states that the pair (x_1,y_1) satisfies the second equation. The proof is now complete.

It is equally easy to show that a solution of the system consisting of equations (2) and (3) is also a solution of (1) and that a solution of the system consisting of (1) and (2) is a solution of (3). We can summarize these results by stating that these three systems are equivalent according to our definition of equivalent systems. (It should be observed that either the system consisting of (1) and (3) or the system consisting of (2) and (3) is a simpler system than the first, since we chose our multipliers in such a way that the equation (3) reduces to an equation in x only, namely $x = 2$.) If we look at the second system

$$\begin{array}{ll} (2) & \begin{cases} 5x - 3y - 25 = 0 \\ \quad x = 2, \end{cases} \\ (3) & \end{array}$$

we can obtain its solution set as follows. Any pair (x,y) which satisfies the second equation has the form $(2,a)$ for some real number a. The pair belongs to the solution set of the system if and only if it also satisfies the first equation; that is

$$10 - 3a - 25 = 0.$$

But this is true if and only if $a = -5$. Hence, the solution set of the system is $\{(2,-5)\}$.

In Example 7-2d, we have used very strongly the fact that the system
$$\begin{cases} 3(3x + 2y + 4) + 2(5x - 3y - 25) = 0 \\ 5x - 3y - 25 = 0 \end{cases}$$
is equivalent to the system
$$\begin{cases} 3x + 2y + 4 = 0 \\ 5x - 3y - 25 = 0. \end{cases}$$
The left member of the first equation of the first system above is called a linear combination of the left members of the equations in the second system.

The same argument which we have used in this example can be used to show that the system
$$f(x,y) = 0$$
$$g(x,y) = 0$$
where $f(x,y)$ and $g(x,y)$ are expressions in the two variables x and y, is equivalent to the system
$$\begin{cases} af(x,y) + bg(x,y) = 0 \\ f(x,y) = 0 \end{cases}$$
or the system
$$\begin{cases} af(x,y) + bg(x,y) = 0 \\ g(x,y) = 0. \end{cases}$$
See Problems 15 and 16.
This general result can be stated in the following principle:

Principle 7-2b. Principle of Linear Combination. The system of equations obtained by setting each of two expressions involving x and y equal to zero is equivalent to the system obtained by pairing either of these expressions with an equation obtained by setting a linear combination of the two expressions equal to zero.

We illustrate the use of this principle in solving systems of equations in the following example.

Example 7-2e. Find the solution set of the system:

$$\begin{cases} 3x - y - 8 = 0 \\ x + 2y - 5 = 0. \end{cases}$$

The system is equivalent to the system

$$3x - y - 8 = 0$$
$$a(3x - y - 8) + b(x + 2y - 5) = 0$$

We may choose a and b in such a way as to eliminate either x or y from the second equation. Let us choose $a = 2$, $b = 1$. the system then becomes

$$\begin{cases} 3x - y - 8 = 0 \\ 7x - 21 = 0 \end{cases}$$

Omitting the details of the proof, we show the remainder of the series of equivalent systems:

$$\begin{cases} 3x - y - 8 = 0 \\ \qquad x - 3 = 0 \end{cases}$$

$$\begin{cases} -y + 1 = 0 \\ x - 3 = 0 \end{cases}$$

$$\begin{cases} y = 1 \\ x = 3 \end{cases}$$

The solution set of the original system is the same as the solution set of the equivalent system

$$\begin{cases} y = 1 \\ x = 3; \end{cases}$$

that is $\{(3,1)\}$.

This method of solving systems of linear equations is essentially the same as the elimination method, which you have probably used many times before. The only real difference is that the definition of equivalent systems and the principle of linear combination assure us that the solution set of the system we obtain in the end is the same as the solution set of the original system.

Exercises 7-2

1. Determine whether or not the following sets of equations are equivalent. Justify your answer.

   (a) $3x = 6$ and $2x = 3$

   (b) $4x + 3y = 12$ and $3x + 4y = 12$

   (c) $5x + 20 = 35$ and $x = 3$

   (d) $8x - 10 = 2y$ and $4x - y = 5$

   (e) $x = y$ and $x^2 + y^2 = 0$

   (f) $x = -\sqrt{y + 3}$ and $x^2 = y + 3$

   (g) $x = \sqrt{y - 6}$ and $x^2 = y - 6$

   (h) $x - 2 = y$ and $|x - 2| = y$

   (i) $y = x^2$ and $y = |x|$

   (j) $xy + x^2 = 0$ and $y = -x$

   (k) $x^2 - 4x - 12 = 0$ and $x = 6$

2. If $(3,5)$ is the only element of the solution set of the system (i) $\begin{cases} 3x + 4y = 29 \\ 3x - 4y = -11, \end{cases}$ is the system (ii) $\begin{cases} 3x + 4y = 29 \\ 6x = 18 \end{cases}$ equivalent if it is known that the solution set of (ii) has only one element?

3. If the solution set of (i) $\begin{cases} x + y = 8 \\ x - y = 4 \end{cases}$ is $\{(6,2)\}$, and (ii) $\begin{cases} 2x - y = 10 \\ 5x + 2y = 34 \end{cases}$ has a single element in its solution set, is (i) equivalent to (ii)?

4. Determine whether or not the following sets of systems are equivalent. Justify your answer.

(a) $\begin{cases} x + y = 10 \\ x - y = 6 \end{cases}$ and $\begin{cases} 2x + 2y = 20 \\ x - y = 6 \end{cases}$

(b) $\begin{cases} 5x + 4y = 3 \\ x + y = 0 \end{cases}$ and $\begin{cases} y = -3 \\ x = 3 \end{cases}$

(c) $\begin{cases} 7x + 3y = 15 \\ 5x - 2y - 19 = 0 \end{cases}$ and $\begin{cases} 7x + 3y = 15 \\ 2(7x+3y-15) + 3(5x-2y-19) = 0 \end{cases}$

(d) $\begin{cases} 3x - 4y = -24 \\ -5x - 3y = 11 \end{cases}$ and $\begin{cases} 4(3x-4y+24) + (5x+3y+11) = 0 \\ 5x + 3y + 11 = 0 \end{cases}$

(e) $\begin{cases} 3x + 5y = 18 \\ 2x + 1 = y \end{cases}$ and $\begin{cases} x = 1 \\ 2x + 1 = y \end{cases}$

(f) $\begin{cases} x^2 + 3y = 6 \\ x + y = 9 \end{cases}$ and $\begin{cases} x^2 + 3y = 6 \\ 3x + 3y = 27 \end{cases}$

(g) $\begin{cases} 5x + 4y - 3 = 0 \\ x + 2y = 0 \end{cases}$ and $\begin{cases} x = 1 \\ y = -\frac{1}{2} \end{cases}$

(h) $\begin{cases} x + y = -8 \\ x^2 - y^2 = 96 \end{cases}$ and $\begin{cases} x^2 - 100 = 0 \\ y^2 = 4 \end{cases}$

(i) $\begin{cases} x^2 + y^2 = 25 \\ y = \frac{1}{4}x \end{cases}$ and $\begin{cases} y = 5 - x \\ y = \frac{1}{4}x \end{cases}$

(j) $\begin{cases} x - y = 0 \\ 2x - 7y = 5 \end{cases}$ and $\begin{cases} x^2 - y^2 = 0 \\ 2x - 7y = 5 \end{cases}$

5. Sketch the graph of the component equations of

(i) $\begin{cases} x^2 + y^2 = 25 \\ y = \frac{4}{3}x \end{cases}$ and (ii) $\begin{cases} x^2 - y^2 = 9 \\ y = \frac{4}{3}x \end{cases}$

Determine from the graphs whether or not the systems are equivalent.

6. Is the system (i) $\begin{cases} x^2 - y^2 = 25 \\ x = y + 5 \end{cases}$ equivalent to

(ii) $\begin{cases} x - y = 5 \\ x + y = 5 \end{cases}$

Sketch the graphs of the component equations to check your answer. Form another system which will be equivalent to (ii). Are all three of these systems equivalent?

7. Sketch the graph of the system (i) $\begin{cases} x^2 - y^2 = 16 \\ x^2 + 4y^2 = 4 \end{cases}$

and the system (ii) $\begin{cases} y = x - 4 \\ x^2 + 4y^2 = 4 \end{cases}$

Use these graphs to help you discuss whether or not these systems are equivalent.

8. Choose a and b, not both zero, in each of the following so as to eliminate the term in y:

(a) $a(y - x^2) + b(y - 2x - 3)$

(b) $a(2x^2 + 7y) + b(3x + 3y - 5)$

(c) $a(3x^2 + 2y - 5) + b(3x^2 - 3y + 7)$

(d) $a(x^2 + 9y + 8) + b(4x^2 - 2y + 7)$

9. Choose a and b, not both zero, in each of the following so as to eliminate the term in x:

   (a) $a(x + y + 3y^2 - 7) + b(y - x)$

   (b) $a(x + 3y - 7) + b(2y - 5x)$

   (c) $a(5x - 7 + 2y) + b(y^2 - 11x + 21)$

   (d) $a(5x - 7 + 2y) + b(y^2 + 11x + 21)$

10. Choose a and b, not both zero, so as to eliminate one of the variables:

    (a) $a(x^2 + y^2 - 7x + 3) + b(2x^2 + 5y^2 - 14x + y)$

    (b) $a(x + 3y - 7x^2 + 2) + b(21x^2 - 9y + y^2 - 3x + 10)$

    (c) $a(x + 3x^2 + 2y + 7) + b(x - 5y + 21)$

    (d) $a(x^2 + 2y^2 + x + 4y - 7) + b(2x^2 + y^2 + x + 2y + 12)$

11. (a) Using the constants $\underline{a}$ and $\underline{b}$ form two systems equivalent to the system

$$\begin{cases} x + y = 1 \\ 2x - y = 4 \end{cases}$$

by the principle of linear combination.

(b) Select several real number values for $\underline{a}$ and $\underline{b}$. Draw the graph of the component equations of the equivalent systems formed on the same coordinate axes.

(c) Select real number values for $\underline{a}$ and for $\underline{b}$ so as to eliminate the term in x; so as to eliminate the term in y. Draw the graph of the component equations of these two systems on the coordinate axes used above.

12. Given the following equivalent systems:

$$\text{(i)}\ \begin{cases} x + y = 2 \\ 2x - 5y = 4 \end{cases} \qquad \text{(ii)}\ \begin{cases} x = 2 \\ x + y = 2 \end{cases} \qquad \text{(iii)}\ \begin{cases} x = 2 \\ y = 0 \end{cases}$$

(a) What real numbers a and b will change (i) to (ii) by the principle of linear combination? (ii) to (iii) by the principle of linear combination?

(b) Sketch a graph on the same coordinate axes of the component equations of these three systems; (i), (ii), and (iii).

(c) Give the solution set of (i), (ii) and (iii).

13. By use of equivalent systems and the principle of linear combination, find the solution set of each of the following systems:

(a) $\begin{cases} 2x - y - 4 = 0 \\ x - 2y + 7 = 0 \end{cases}$

(b) $\begin{cases} 7x + 5y = 11 \\ 3x - 2y = 13 \end{cases}$

(c) $\begin{cases} .02y = .01x - .1 \\ .03x - .1y = 0 \end{cases}$

(d) $\begin{cases} \frac{2}{3}x + \frac{4}{3}y = 4 \\ x + 2y = 4 \end{cases}$

(e) $\begin{cases} 11x + 3y + 7 = 0 \\ 2x + 5y = 21 \end{cases}$

(f) $\begin{cases} 2y = 2x - 1 \\ x = 2y - 2 \end{cases}$

(g) $\begin{cases} 2x - 3y = 5 \\ x - 1.5y = 2.5 \end{cases}$

(h) $\begin{cases} 2x = 8y - 10 \\ 5 = \frac{1}{3}y - x \end{cases}$

(i) $\begin{cases} 0 = -\frac{2}{3}x - \frac{1}{3}y - \frac{5}{3} \\ x = \frac{5}{4}y + 1 \end{cases}$

(j) $\begin{cases} \frac{1}{2}x + \frac{3}{4}y = 1 \\ x - \frac{1}{3}y = \frac{7}{3} \end{cases}$

*14. Prove that if system (1) is equivalent to system (2), and if system (2) is equivalent to system (3), then system (1) is equivalent to system (3).

*15. Prove that the system $\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$ is equivalent to the system $\begin{cases} a\cdot f(x,y) + b\cdot g(x,y) = 0 \\ g(x,y) = 0 \end{cases}$

*16. If $f(x,y)$ and $g(x,y)$ are algebraic expressions, show that the systems,

$$\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases} \quad \text{and} \quad \begin{cases} a\cdot f(x,y) + b\cdot g(x,y) = 0 \\ c\cdot f(x,y) + d\cdot g(x,y) = 0 \end{cases} \quad \text{are}$$

equivalent if and only if $ad - bc \neq 0$

7-3. Systems of Linear Equations.

In this section we are concerned with finding the solution set of the system

7-3a $$\begin{cases} a_1x + b_1y + c_1 = 0, & \text{not both } a_1 \text{ and } b_1 \text{ zero} \\ a_2x + b_2y + c_2 = 0, & \text{not both } a_2 \text{ and } b_2 \text{ zero.} \end{cases}$$

We now have several ways of attacking this problem. The method of eliminating one of the variables, as we have seen in the preceding section, is essentially the same thing as finding an equivalent system using the principle of linear combination.

In addition, we may consider the problem from the geometric point of view. The machinery of analytic geometry which we developed in Chapter 6 will be extremely useful in this method of solution.

We begin by considering some examples.

Example 7-3a. Find the solution set of the system

$$\begin{cases} x + y - 1 = 0 \\ 2x = 2 - 2y. \end{cases}$$

Solution: The system is equivalent to the system

$$\begin{cases} x + y - 1 = 0 \\ 2x + 2y - 2 = 0. \end{cases}$$

But the left member of the second equation is simply twice the left member of the first equation. Hence, any ordered pair which satisfies the first equation will satisfy the second. The system is dependent and the solution set of the system is the set of points on the line whose equation $x + y - 1 = 0$; that is, $\{(a,1-a)\}$, where a is any real number. Geometrically the two equations are equations for the same straight line.

Example 7-3b. Find the solution set of the system

$$\begin{cases} x + y + 1 = 0 \\ 2x + 2y + 1 = 0. \end{cases}$$

Solution: The system is equivalent to the system

$$\begin{cases} 2(x + y + 1) - (2x + 2y + 1) = 0 \\ \qquad\qquad\qquad x + y + 1 = 0 \end{cases}$$

which is equivalent to
$$\begin{cases} 1 = 0 \\ x + y + 1 = 0. \end{cases}$$

It is clear that there are no ordered pairs (x,y) which satisfy the equation $1 = 0$. And since the system

$$\begin{cases} 1 = 0 \\ x + y + 1 = 0 \end{cases}$$

is equivalent to the original system, the solution set of the original system is the empty set. Hence, the system is inconsistent. Geometrically the lines must be parallel. This follows since the two lines have the same slope, -1, but not the same y-intercepts.

Example 7-3c. Find the solution set of the system

$$\begin{cases} 2x + 3y + 1 = 0 \\ 3x - 5y + 4 = 0. \end{cases}$$

Solution: The system is equivalent to

$$\begin{cases} -3(2x + 3y + 1) + 2(3x - 5y + 4) = 0 \\ 2x + 3y + 1 = 0, \end{cases}$$

that is $\begin{cases} 19y - 5 = 0 \\ 2x + 3y + 1 = 0, \end{cases}$ which is equivalent to $\begin{cases} y = \frac{5}{19} \\ 2x + 3y + 1 = 0 \end{cases}$

which is equivalent to $\begin{cases} y = \frac{5}{19} \\ x = -\frac{17}{19}. \end{cases}$

The solution set is therefore $\{(-\frac{17}{19}, \frac{5}{19})\}$, and *the system is consistent*. Geometrically the lines intersect in the point $(-\frac{17}{19}, \frac{5}{19})$.

We return now to the general system 7-3a,

$a_1x + b_1y + c_1 = 0$, not both a_1 and b_1 zero.

$a_2x + b_2y + c_2 = 0$, not both a_2 and b_2 zero.

The graphs of the two equations of this system are straight lines (Section 6-2). Let us call them L_1 and L_2. Geometrically, three cases are possible.

Case I. The lines L_1 and L_2 are the same line.

Case II. The lines L_1 and L_2 are parallel.

Case III. The lines L_1 and L_2 intersect in a single point.

Case I. We have noted already in Chapter 6 that the graphs of the two equations are the same straight line if and only if the corresponding coefficients are proportional; that is, $a_1 = ka_2$, $b_1 = kb_2$, $c_1 = kc_2$. ($k \neq 0$. Why not?) In this case *the system is dependent*.

Case II. We also noted in Chapter 6 that two distinct lines are parallel if and only if they have the same slope (or are both vertical). Since the slopes are

$$m_1 = -\frac{a_1}{b_1} \text{ and } m_2 = -\frac{a_2}{b_2},$$

the lines are parallel if and only if

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} \text{ or } a_1b_2 = a_2b_1.$$

If the lines are vertical, $b_1 = b_2 = 0$ and $a_1b_2 = a_2b_1 = 0$.

Therefore two distinct lines are parallel if and only if

$$a_1b_2 = a_2b_1 \text{ or } a_1b_2 - a_2b_1 = 0.$$

In this case the system is inconsistent.

Case III. We shall show that two lines intersect in a single point (and are therefore consistent) if and only if

$$a_1b_2 - a_2b_1 \neq 0.$$

Using the principle of linear combination the original system is equivalent to the system

$$\begin{cases} b_2(a_1x + b_1y + c_1) - b_1(a_2x + b_2y + c_2) = 0 \\ a_1x + b_1y + c_1 = 0, \end{cases}$$

which is equivalent to the system

$$\begin{cases} x = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \\ a_1x + b_1y + c_1 = 0, \end{cases}$$

which is equivalent to the system

$$\begin{cases} x = \dfrac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1} \\[2ex] y = \dfrac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1} . \end{cases}$$

The solution set of the original system is the same as that of the last system and this is clearly

$$\left\{\left(\frac{b_1c_2 - b_2c_1}{a_1b_2 - a_2b_1}, \frac{a_2c_1 - a_1c_2}{a_1b_2 - a_2b_1}\right)\right\} .$$

Hence, the two lines intersect in this single point if and only if $a_1b_2 - a_2b_1 \neq 0$, since the systems are equivalent if and only if $a_1b_2 - a_2b_1 \neq 0$.

Example 7-3d. Is the system $\begin{cases} 5x + 4y + 7 = 0 \\ 2x - 7y + 5 = 0 \end{cases}$ consistent?

Solution: Since $a_1b_2 - a_2b_1 = 5(-7) - (4)(2) = -43 \neq 0$, the system is consistent. The solution set is the single number pair, $\left(\frac{-69}{43}, \frac{11}{43}\right)$.

Let us look again at the system $\begin{cases} a_1x + b_1y + c_1 = 0 \\ a_2x + b_2y + c_2 = 0. \end{cases}$

If we consider the equation obtained by setting the linear combination $k_1(a_1x + b_1y + c_1) + k_2(a_2x + b_2y + c_2) = 0$, the result is again a linear equation. Its graph, therefore, is a straight line by Section 6-2. Furthermore, if the two given lines intersect in $Q(x_o,y_o)$, then $a_1x_o + b_1y_o + c_1 = 0$ and $a_2x_o + b_2y_o + c_2 = 0$ and for any real numbers k_1 and k_2, $k_1(a_1x_o + b_1y_o + c_1) + k_2(a_2x_o + b_2y_o + c_2) = k_1\cdot 0 + k_2\cdot 0 = 0$. Therefore, $Q(x_o,y_o)$ is on the line. So our Principle 7-2b simply asserts that the system of equations whose graphs are the two lines intersecting in $Q(x_o,y_o)$ is equivalent to the system consisting of one of these lines and any other line which passes through $Q(x_o,y_o)$. Our method amounts to finding the equivalent system consisting of the horizontal and vertical lines passing through $Q(x_o,y_o)$; that is, the system

$$x = x_o$$
$$y = y_o.$$

Example 7-3e. The system $\begin{cases} x + 3y = 9 \\ x - 3y = -3 \end{cases}$ is equivalent to the system $\begin{cases} k_1(x + 3y - 9) + k_2(x - 3y + 3) = 0 \\ x - 3y + 3 = 0 \end{cases}$

Any line through the point of intersection of the lines of the original system can be represented by the first equation in the second system for some values of k_1 and k_2. In particular, if $k_1 = 1$ and $k_2 = -1$ the resulting equation is the equation of the vertical line $3y - 9 + 3y - 3 = 0$; that is $y = 2$. If $k_1 = 1$ and $k_2 = 1$, the resulting equation is the horizontal line $x = 3$.

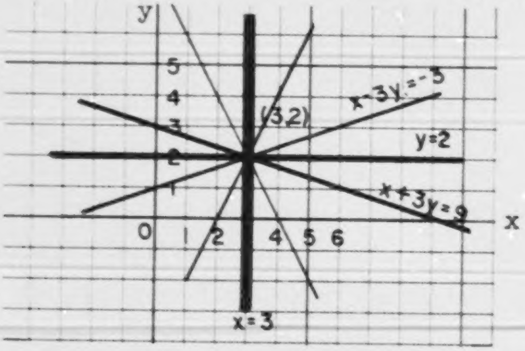


Fig. 7-3a.

Exercises 7-3

1. Tell whether the graphs of the component equations of each of the following systems are the same straight line, parallel lines, or intersecting lines. Also, tell whether these systems are consistent, inconsistent, or dependent?

(a) $\begin{cases} 5x + 4y + 7 = 0 \\ 2x - 7y + 5 = 0 \end{cases}$

(b) $\begin{cases} 3x + 3y + 1 = 0 \\ 2x + 2y + 1 = 0 \end{cases}$

(c) $\begin{cases} 3x = 1 - 2y \\ \frac{9}{2}x - 6y = 3 \end{cases}$

(d) $\begin{cases} .2x - .5y = .1 \\ .4x = y - .2 \end{cases}$

(e) $\begin{cases} y = \frac{2}{5}x - 1 \\ y = \frac{9}{5}x + 6 \end{cases}$

(f) $\begin{cases} 2x - y = -3 \\ 9 = x + y \end{cases}$

(g) $\begin{cases} y = \frac{1}{3}x + 5 \\ y = \frac{1}{3}x - 5 \end{cases}$

(h) $\begin{cases} 2x - y = 6 \\ 4x - 2y = 5 \end{cases}$

(i) $\begin{cases} 3x - 2y = 1 \\ 6x - 4y = 2 \end{cases}$

(j) $\begin{cases} \frac{x}{3} + \frac{y}{5} = 1 \\ 10x + 6y = 5 \end{cases}$

2. Find the solution sets of the following systems:

(a) $\begin{cases} x + 3y = 9 \\ x - 3y = -3 \end{cases}$

(b) $\begin{cases} 4x + y = 5 \\ 2x - 3y = 13 \end{cases}$

(c) $\begin{cases} 2x - 9y = 5 \\ 3x - 3y = 11 \end{cases}$

(d) $\begin{cases} 3x - 7y = 1 \\ 2x - 3y = -1 \end{cases}$

(i) $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y + 2 = 0 \end{cases}$

(j) $\begin{cases} 2x + 2y = 100 \\ \frac{x}{2} + \frac{3y}{5} = 14 \end{cases}$

(k) $\begin{cases} \frac{3x + 1}{5} = \frac{3y + 2}{4} \\ \frac{2x - 1}{5} + \frac{3y - 2}{4} = 2 \end{cases}$

(ℓ) $\begin{cases} x + 2y - 3 = 0 \\ 12 = 8y + 4x \end{cases}$

(e) $\begin{cases} 4x + y = 2 \\ 2x - 3y = 8 \end{cases}$

(f) $\begin{cases} 3x = -3y - 4 \\ x - 6y = 3\frac{1}{3} \end{cases}$

(g) $\begin{cases} 3x + 4y = 16 \\ 5x + 3y = 12 \end{cases}$

(h) $\begin{cases} \frac{x}{3} + \frac{y}{6} = \frac{2}{3} \\ \frac{2x}{5} + \frac{y}{4} = \frac{1}{5} \end{cases}$

(m) $\begin{cases} \frac{x - 5}{4} = \frac{6y}{8} \\ 2x - 3y = 5 \end{cases}$

(n) $\begin{cases} \frac{x - y}{2} - \frac{x - 4y}{6} = 4 \\ \frac{x + y}{9} - \frac{x - 2y}{6} = \frac{22}{9} \end{cases}$

(o) $\begin{cases} \frac{x}{a} - \frac{y}{b} = \frac{c}{ab} \\ \frac{x}{a} + \frac{y}{b} = \frac{d}{ab} \end{cases}$

(p) $\begin{cases} x + ay = b \\ 2x - by = a \end{cases}$

*3. Prove that if $a_1b_2 - a_2b_1 = 0$, $b_1c_2 - b_2c_1 = 0$, and $a_2c_1 - a_1c_2 = 0$, then there exists a real number $k \neq 0$ such that $a_1 = ka_2$, $b_1 = kb_2$, and $c_1 = kc_2$. Assume, of course, that $a_1b_1 \neq 0$ and $a_2b_2 \neq 0$.

4. A man can row downstream 6 miles in 1 hour and return in 2 hours. Find his rate in still water and the rate of the river.

5. If a field is enlarged by making it 10 rods longer and 5 rods wider, its area is increased by 1050 square rods. If its length is decreased by 5 rods and its width is decreased by 10 rods, its area is decreased by 1050 square rods. Find the original dimensions of the field.

6. The sum of the acute angles of an obtuse triangle is 85°. If the difference of the acute angles is 19°, what are the angles?

7. A and B are 30 miles apart. If they leave at the same time and travel in the same direction, A overtakes B in 8 hours. If they walk toward each other, they meet in 3 hours. What are their rates?

8. One alloy contains 3 times as much copper as silver, another contains 5 times as much silver as copper. How much of each alloy must be used to make 14 pounds in which there is twice as much copper as silver?

9. Find two numbers such that

   (a) Their sum is 12 and their difference is 3;

   (b) The sum of their reciprocals is 24 and the difference of their reciprocals is 4.

10. The formula $s = s_o + v_o t - 16\ t^2$ is often used for falling bodies where s is the height of the body at any time t, s_o is the initial height (when $t = 0$), v_o is the initial velocity and the coefficient 16 is used for one half of the acceleration of gravity. Distance s is in feet and time t is in seconds. If $s = 10{,}000$ when $t = 5$, and $s = 8{,}550$ when $t = 10$, find s_o and v_o.

11. Find an equation of the line which passes through the origin and the intersection of the lines whose equations are $4x + y = 2$ and $2x - 3y = 8$.

12. Find an equation of the line which passes through the point $(5,4)$ and the intersection of the lines whose equations are $y = -\frac{1}{4}x + \frac{1}{2}$ and $x + \frac{3}{2}y = -\frac{1}{2}$.

7-4. Systems of One Linear and One Quadratic Equation.

The simplest kind of system of two equations in which at least one is not linear, is a system consisting one linear and one quadratic equation.

Example 7-4a. Find the solution set of the system

$$\begin{cases} y = x^2 \\ y = 2x + 3. \end{cases}$$

Solution: If (x,y) belongs to the solution set of the system, then it must have the form $(a, 2a + 3)$ for some real number a, in order to belong to the solution set of the second equation. On the other hand, to belong to the solution set of the first, the ordered pair must have the form (a, a^2) for some real number a. Hence, a pair with first element a, belongs to the solution set of the system if and only if $2a + 3 = a^2$. This implies that $a = -1$ or $a = 3$. Hence, the solution set of the system is $\{(-1,1), (3,9)\}$.

Example 7-4b. Find the solution set of the system

$$\begin{cases} x^2 + y^2 = 5 \\ x + 2y = 5. \end{cases}$$

Solution: The elements of the solution set of the equation $x + 2y = 5$ must have the form $(a, \frac{5-a}{2})$. The pair will satisfy $x^2 + y^2 = 5$ in addition, if and only if

$$a^2 + \left(\frac{5-a}{2}\right)^2 = 5$$

$$4a^2 + 25 - 10a + a^2 = 20$$

$$5a^2 - 10a + 5 = 0$$

$$a^2 - 2a + 1 = 0$$

$$(a - 1)^2 = 0$$

Hence, $a = 1$ and the solution set of the system is $\{(1,2)\}$.

Example 7-4c. Find the solution set of the system

$$\begin{cases} x^2 - 4y^2 = 1 \\ x = y. \end{cases}$$

Solution: The elements of the solution set of the second equation must have the form (a,a). The pair will satisfy the first equation also, if and only if

$$a^2 - 4a^2 = 1$$

$$-3a^2 = 1$$

$$a^2 = -\frac{1}{3}$$

But this equation is not satisfied by any real number a. Hence, the solution set of the system is the empty set.*

Example 7-4d. Find the solution set of the system

$$\begin{cases} x^2 - y^2 = 0 \\ x - y = 0 \end{cases}$$

Solution: The elements of the solution set of the second equation must have the form (a,a). The pair will satisfy the first equation also, if and only if,

$$a^2 - a^2 = 0.$$

Since this equation is satisfied by every real number a, its solution set is the set of all real numbers. Therefore, the solution set of the original system is the set of all pairs (a,a) where a is any real number.

The preceding examples exhibit four different kinds of solution sets for this kind of system, namely, the empty set, a set consisting of only one pair of real numbers, a set consisting of two pairs of real numbers, and a set whose graph is a certain line. Moreover, these are the only kinds of solution sets which can occur, as we proceed to show.

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* Note that if we allowed the variables to represent complex numbers our solution set would be:

$$\{(\frac{i\sqrt{3}}{3}, \frac{i\sqrt{3}}{3}), (-\frac{i\sqrt{3}}{3}, -\frac{i\sqrt{3}}{3})\}.$$

Suppose we wish to find the solution set of the system

$$\text{(i)} \qquad \begin{cases} Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \\ Lx + My + N = 0, \end{cases}$$

where not all of A, B, C, D, or E are 0, and where M is not 0. Any member of the solution set of $Lx + My + N = 0$ must have the form $(a, -\frac{La + N}{M})$. The pair belongs to the solution set of the system if and only if it satisfies, in addition, the first equation of the system, that is

(ii)

$$Aa^2 + Ba(-\frac{La + N}{M}) + C(-\frac{La + N}{M})^2 + Da + E(-\frac{La + N}{M}) + F = 0.$$

This equation can be expressed as,

$$\text{(iii)} \qquad A_0a^2 + B_0a + C_0 = 0.$$

If $A_0 = B_0 = C_0 = 0$, every real number a satisfies the equation.

If $A_0 = B_0 = 0$, but $C_0 \neq 0$, no real number a satisfies the equation.

If $A_0 = 0$ but $B_0 \neq 0$, there is one real number a which satisfies the equation.

If $A_0 \neq 0$, there are either no, one, or two real numbers for a which satisfy the equation.

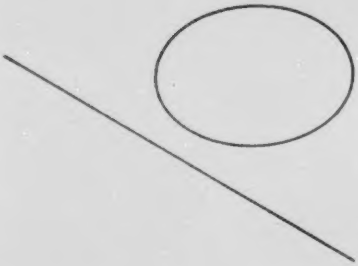
This result has a very interesting geometric interpretation. It means that any straight line

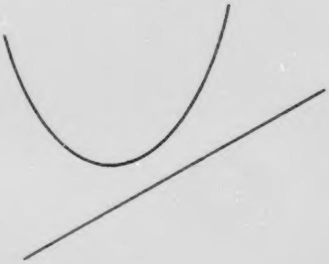
(1) will not intersect a conic or
(2) will intersect it once or
(3) will intersect it twice or
(4) will actually be a part of the graph of the conic.

The fourth case can occur when the conic is "degenerate" in the sense that its equation can be expresses as the product of two linear factors one of which is the linear equation of the system as in Example 7-4d above. In this case, the graph of the linear equation is actually a part of the graph of the quadratic equation.

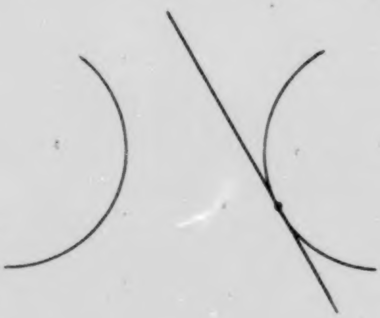
Some graphical interpretations of statements, 1-4 are shown in the following sketches:

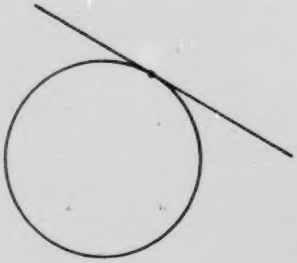
(1) The line does not intersect the conic.



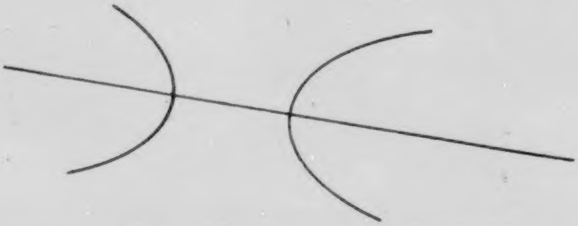


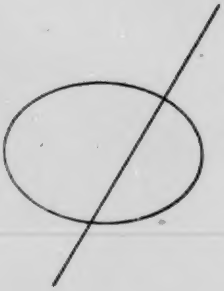
(2) The line intersects the conic once.





(3) The line intersects the conic twice.





(4) The line is a part of the graph of a degenerate conic whose graph consists of two (intersecting, parallel, or coincident) lines.

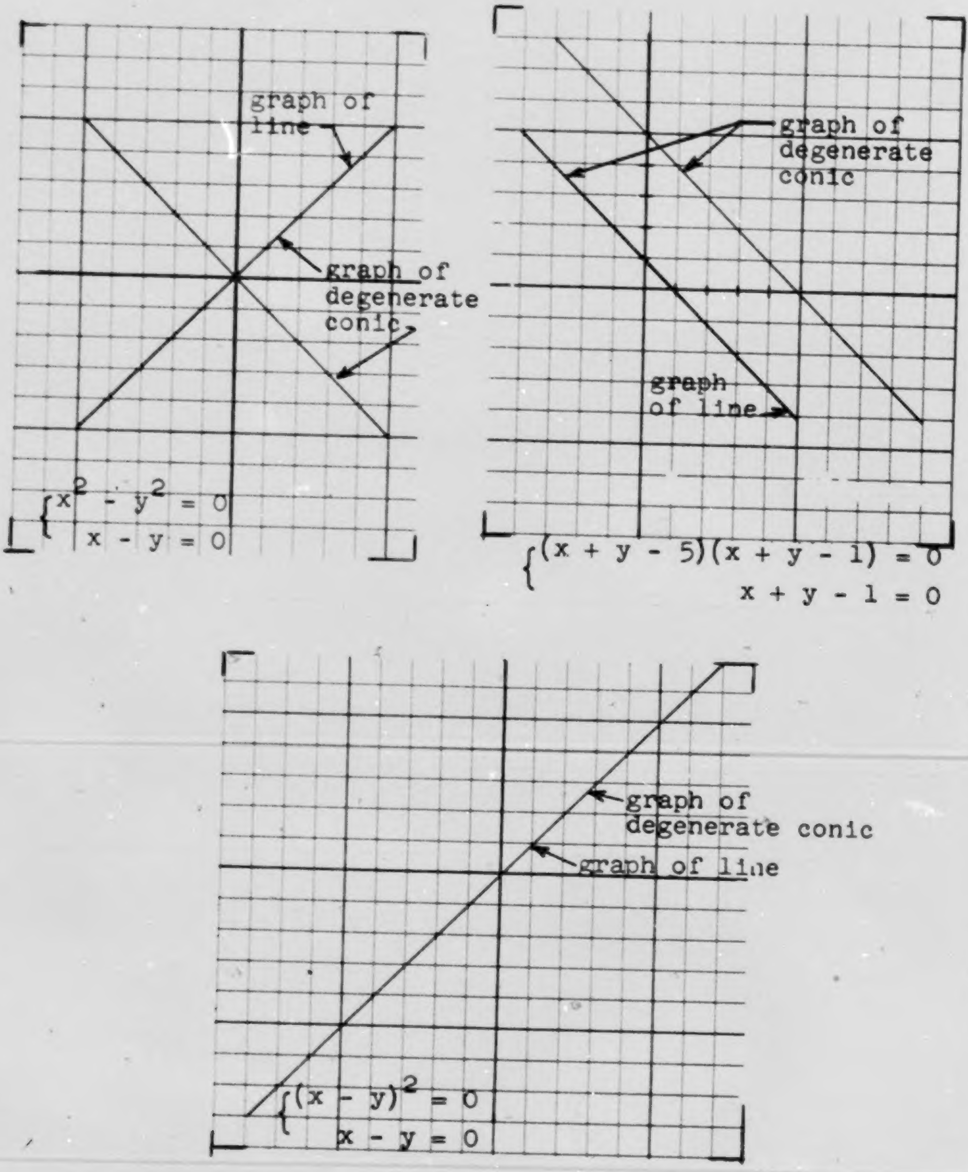


Fig. 7-4b

The previous discussion eliminates the possibility mentioned in Chapter 4 that a parabola might actually look like the curve in Fig. 7-4c.

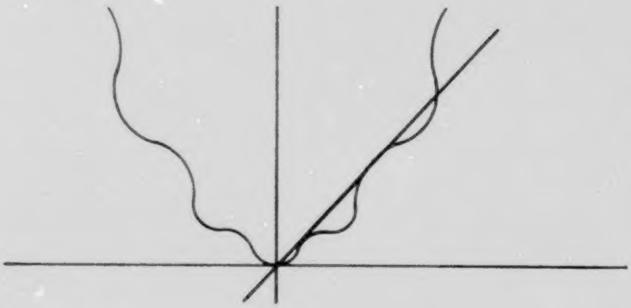


Fig. 7-4c.

For, if one such wiggle occurred, we could draw a line intersecting the parabola in three points.

In the examples we have considered so far, we found the solution set of the system by first determining the form which a number pair with first element a must have if it is to satisfy the linear equation. Then we reasoned that the number pair belongs to the solution set of the system if and only if it also satisfies the quadratic equation. This transformed the problem of finding the solution set of the system into the problem of solving a quadratic equation. Of course, we might just as well have said suppose the second element of the ordered pair is b; then if the pair is to satisfy the linear equation, the first element of the pair must have a certain form, etc. In some systems this approach greatly simplifies the algebraic manipulations involved in finding the solution set.

Example 7-4e. Find the solution set of the system

$$\begin{cases} 2y^2 + xy = 5 \\ x + 4y = 7. \end{cases}$$

Solution: If an ordered pair whose second member is b, belongs to the solution set of the second equation, then the ordered pair must have the form $(7-4b,b)$. The pair is an element of the solution set of the system if and only if it satisfies the first equation. That is,

$$2b^2 + (7 - 4b)b = 5$$

$$2b^2 + 7b - 4b^2 = 5$$

$$2b^2 - 7b + 5 = 0$$

$$(2b - 5)(b - 1) = 0.$$

Hence, the ordered pair $(7 - 4b,b)$ belongs to the solution set of the system if and only if $b = 1$ or $b = \frac{5}{2}$. The solution set of the system is $\{(3,1), (-3, \frac{5}{2})\}$.

Exercises 7-4

1. Find the solution set of each of the following systems. Use the procedure developed in this section.

(a) $\begin{cases} x^2 + y^2 = 50 \\ x - y = 0 \end{cases}$

(b) $\begin{cases} x^2 - 4x + 3 = 0 \\ x - y + 1 = 0 \end{cases}$

(c) $\begin{cases} x^2 + 4y + 2x - 11 = 0 \\ y = x + 5 \end{cases}$

(d) $\begin{cases} x^2 - y^2 = 0 \\ x + y = 0 \end{cases}$

(g) $\begin{cases} y^2 + 3x + y = 7 \\ x + 10 = y \end{cases}$

(h) $\begin{cases} 3x - 2y = 0 \\ x^2 + y^2 = 52 \end{cases}$

(i) $\begin{cases} xy = -12 \\ x + 14 = 2y \end{cases}$

(j) $\begin{cases} 2x^2 - xy = y^2 \\ x = y \end{cases}$

(e) $\begin{cases} xy = 6 \\ 2x - y = 1 \end{cases}$

(f) $\begin{cases} y = 2x^2 \\ y + 1 = 2x \end{cases}$

(k) $\begin{cases} 3x^2 - y^2 = 3 \\ 2y - x = 8 \end{cases}$

(ℓ) $\begin{cases} x^2 - y^2 + x + y = 0 \\ x + 1 = y \end{cases}$

(m) $\begin{cases} y = x^2 - 1 \\ y = - 4x - 5 \end{cases}$

2. Find the solution set of each of the following systems. Check by sketching the graph of the equations of each system.

   (a) $\begin{cases} y = x^2 \\ 2x - y = - 3 \end{cases}$

   (b) $\begin{cases} xy = 9 \\ x + y = 5 \end{cases}$

   (c) $\begin{cases} x^2 + 4y^2 = 25 \\ y - 2 = - \frac{3}{8}(x - 3) \end{cases}$

   (d) $\begin{cases} xy - 2x + 2y + 4 = 0 \\ x - 2 = 0 \end{cases}$

3. Discuss the geometric interpretation of the solution sets of the systems in Problem 1.

4. A line passing through the point $(0,-5)$ is tangent to the conic whose equation is $x^2 = y + 3$. Write an equation of the line. How many tangents are possible? Give the equation of each.

5. A line having slope 2 is tangent to the circle whose equation is $x^2 + y^2 = 16$. Write an equation of this line. How many tangents are possible? Give the equation of each.

*6. Find value of k in terms of r and m so that the line whose equation is $y = mx + k$ will be tangent to the circle whose equation is $x^2 + y^2 = r^2$, $r > 0$.

7-5 Other Systems.

Finding solution sets for systems of equations in which neither component equation is linear, is complicated. There are several special methods which solve the problem for particular types of systems consisting of two quadratic equations. These methods usually consist of finding simpler systems which are equivalent to the original system by eliminating one of the variables from one equation. This elimination process may be essentially our method of linear combination, or it may involve substituting an expression for one variable obtained from a first equation in a second equation.

Example 7-5a. Find the solution set of the system:

$$\begin{cases} 3x^2 - y^2 + 22 = 0 \\ x^2 + 2y^2 - 107 = 0 \end{cases}$$

Solution: We form a linear combination of the left members which will eliminate y^2, namely

$$2(3x^2 - y^2 + 22) + (x^2 + 2y^2 - 107) = 0.$$

By Principle 7-2b the new system

$$\begin{cases} 3x^2 - y^2 + 22 = 0 \\ 7x^2 - 63 = 0 \end{cases}$$

is equivalent to the original system. This in turn is equivalent to the system

$$\begin{cases} 3x^2 - y^2 + 22 = 0 \\ x^2 - 9 = 0, \end{cases}$$

which is equivalent to

$$\begin{cases} (3x^2 - y^2 + 22) - 3\,(x^2 - 9) = 0 \\ \qquad\qquad\qquad\qquad x^2 - 9 = 0 \end{cases}$$

or

$$\begin{cases} y^2 - 49 = 0 \\ x^2 - \ \ 9 = 0. \end{cases}$$

Just as in the previous section, we may observe that any ordered pair satisfying the first equation must have the form (a,7) or (a,-7) for some real number a; while any pair satisfying the second must have the form (3,b) or (-3,b) for some real number b. Hence, the only ordered pairs satisfying both equations are (3,7), (3,-7), (-3,7), and (-3,-7). Since the system

$$\begin{cases} y^2 - 49 = 0 \\ x^2 - 9 \ = 0 \end{cases}$$

is equivalent to the original system, the solution set of the original system is

$$\{(3,7),\ (3,-7),\ (-3,7),\ (-3-7)\}.$$

Example 7-5b. Find the solution set of the system

$$\begin{cases} 2xy - y^2 + 24 = 0 \\ 2x^2 + xy + 2 \ = 0. \end{cases}$$

By the Principle of Linear Combination (Principle 7-2b) the system is equivalent to the system

$$\begin{cases} -(2xy - y^2 + 24) + 2(2x^2 + xy + 2) = 0 \\ \qquad\qquad\qquad\qquad 2x^2 + xy + 2 \ = 0; \end{cases}$$

That is,

$$\begin{cases} 4x^2 + y^2 = 20 \\ 2x^2 + xy = -2. \end{cases}$$

Since no ordered pair with first element zero satisfies the second equation, this system is equivalent to the system

$$\begin{cases} 4x^2 + y^2 = 20 \\ y = -\frac{2}{x} - 2x. \end{cases}$$

As in the last section we now say that if an ordered pair with first element a satisfies the second equation, it must have the form

$$(a, -\frac{2}{a} - 2a).$$

It is a member of the solution set of the system if and only if it is a member of the solution set of the first equation; that is,

$$4a^2 + (-\frac{2}{a} - 2a)^2 = 20.$$

Then since $a \neq 0$ $\quad 4a^2 + \frac{4}{a^2} + 8 + 4a^2 = 20$

$$8a^4 + 4 - 12a^2 = 0$$

$$2a^4 - 3a^2 + 1 = 0$$

$$(2a^2 - 1)(a^2 - 1) = 0$$

Hence, the pair $(a, -\frac{2}{a} - 2a)$ belongs to the solution set of the system if and only if $a = 1$, $a = -1$, $a = \frac{1}{\sqrt{2}}$, or $a = -\frac{1}{\sqrt{2}}$.

Hence, the solution set is

$$\{(1, -4), (-1, 4), (\frac{1}{\sqrt{2}}, -3\sqrt{2}), (-\frac{1}{\sqrt{2}}, 3\sqrt{2})\}.$$

Example 7-5c. Find the solution set of the system

$$\begin{cases} x^2 - y^2 = 0 \\ 2x^2 + xy = 48. \end{cases}$$

Solution: The system can be written

$$\begin{cases} (x + y)(x - y) = 0 \\ 2x^2 + xy = 48. \end{cases}$$

Any ordered pair with first element a satisfying the first equation must either have the form (a,a) or the form (a,-a) for some real number a. If the ordered pair is to be a member of the solution set of the system it must, in addition, satisfy the second equation; that is either

$$\begin{aligned} 2(a)^2 + a(a) &= 48 \\ 3a^2 &= 48 \\ a^2 &= 16 \\ a = 4 \text{ or } a &= -4 \end{aligned}$$

or

$$\begin{aligned} 2(-a)^2 + a(-a) &= 48 \\ a^2 &= 48 \end{aligned}$$

$$a = 4\sqrt{3} \text{ or } a = -4\sqrt{3}$$

Hence, an ordered pair of the form (a,a) is a member of the solution set of the system if and only if a is 4 or -4. A pair of the form (a,-a) is a member of the solution set of the system if and only if $a = 4\sqrt{3}$ or $a = -4\sqrt{3}$. Hence, the solution set of the system is

$$\{(4,4),\ (-4,-4),\ (4\sqrt{3},\ -4\sqrt{3}),\ (-4\sqrt{3},\ 4\sqrt{3})\}.$$

These examples illustrate some of the types of systems for which the solution sets can be found using the methods of this chapter. Of course, not every system can be solved so easily.

Example 7-5d. Find the solution set of the system

$$\begin{cases} x^2 - 4y^2 + 8y - 8 = 0 \\ x^2 + 9y^2 - 4x - 32 = 0. \end{cases}$$

Solution: No linear combination of the left members of the two equations will eliminate either x or y. However, we can eliminate x^2 and obtain the equivalent system

$$\begin{cases}(x^2 - 4y^2 + 8y - 8) - (x^2 + 9y^2 - 4x - 32) = 0\\ x^2 - 4y^2 + 8y - 8 = 0;\end{cases}$$

that is,

$$\begin{cases}-13y^2 + 8y + 4x + 24 = 0\\ x^2 - 4y^2 + 8y - 8 = 0\end{cases}$$

Then while it is possible to use the technique which we have used before of letting b be the second element of an ordered pair which satisfies the first equation, we run into a few complications. The pair must then have the form

$$(\tfrac{13}{4}b^2 - 2b - 6, b).$$

The pair belongs to the solution set of the system, if and only if, in addition it satisfies the second equation; that is,

$$(\tfrac{13}{4}b^2 - 2b - 6)^2 - 4b^2 + 8b - 8 = 0.$$

This equation is an equation of the fourth degree in b and we do not have available methods for solving such equations. So that while in theory our method still applies, in actual practice, we are unable to carry it through successfully. In such situations the number of members of the solution set of the system and approximations for these number pairs can frequently be obtained from the graphs of the component equations. See Fig. 7-5a.

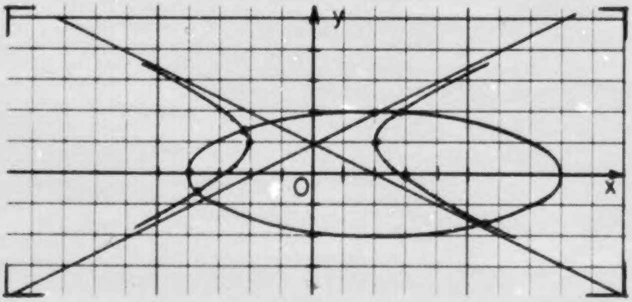


Fig. 7-5a.

The equations may be rewritten in the form

$$\begin{cases} \frac{x^2}{4} - \frac{(y - 1)^2}{1} = 1 \\ \frac{(x - 2)^2}{36} + \frac{y^2}{4} = 1. \end{cases}$$

From Figure 7-5a we see that the solution set of the system contains four number pairs. They are approximately

$$\{(2.8,1.9), \quad (-2.2,1.3), \quad (-3.8,-.7), \quad \text{and } (5.6, -1.6)\}.$$

Solution set for systems of inequalities can be obtained graphically in a similar way.

Example 7-5e. Find the solution set of the system

$$\begin{cases} y < -x^2 + 8x \\ y > x^2 - 12x + 32. \end{cases}$$

Solution: We first sketch the graph the two parabolas whose equations are $y = -x^2 + 8x$ and $y = x^2 - 12x + 32$. We may rewrite the equations $y - 16 = -(x - 4)^2$ and $y - (-4) = (x - 6)^2$. The first is a Parabola with vertex $V_1(4,16)$ and axis the line $x = 4$ which opens downward. The second is the parabola with Vertex $V_2(6,-4)$ and axis the line $x = 6$ opening upwards. The solution set is the set of points below the first parabola and above the second; that is, the shaded region R in Fig. 7-5b, not including points on the boundary.

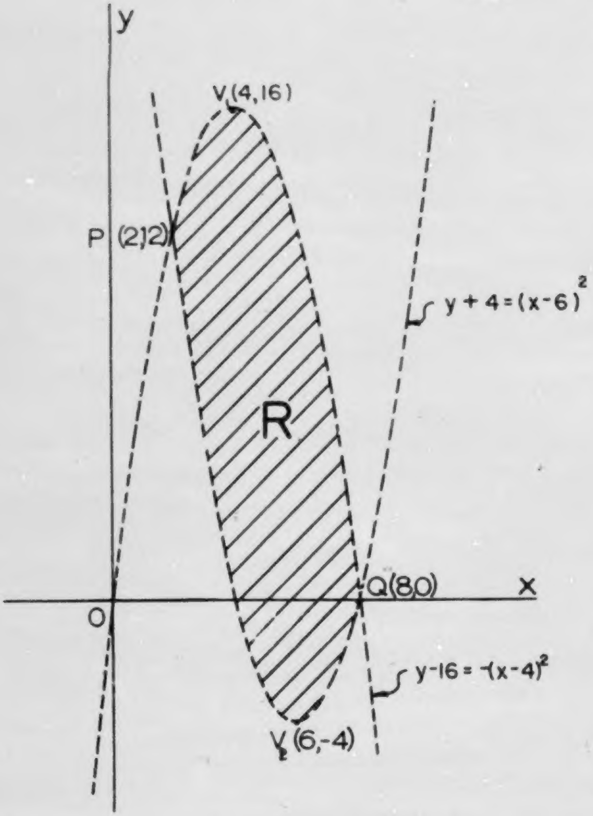


Fig. 7-5b.

Exercises 7-5

1. Use the principle of linear combination to find the solution set of the system

$$\begin{cases} 4x^2 + y^2 = 100 \\ 4x - y^2 = -20. \end{cases}$$

Check your answer by sketching the graph of the component equations of the system.

2. Find the solution set of the system

$$\begin{cases} x^2 + 4y^2 = 100 \\ 2y^2 = 50 - \frac{1}{2}x^2. \end{cases}$$

Give the geometric interpretation of the solution set of this system.

3. Find the solution set of the system

$$\begin{cases} x^2 + y^2 = 20 \\ \frac{1}{2}x^2 = 3 - \frac{1}{2}y^2 \end{cases}$$

Give the geometric interpretation of the solution set of this system.

4. Find the solution set of the system

$$\begin{cases} x^2 + y^2 - 25 = 0 \\ y^2 - x^2 - 2x - 1 = 0 \end{cases}$$

Give the geometric interpretation of the solution set of this system.

5. Find the solution set of the system

$$\begin{cases} x^2 - 5xy + 4y^2 = 0 \\ xy = 1 \end{cases}$$

6. Find the solution set of each of the following systems:

(a) $\begin{cases} x^2 - y^2 = 4 \\ x^2 + y^2 = 16 \end{cases}$

(h) $\begin{cases} x^2 + 4y^2 = 20 \\ x^2 - 5xy + 4y^2 = 0 \end{cases}$

(b) $\begin{cases} x^2 - 7y = 2 \\ x^2 - y^2 = 12 \end{cases}$

(i) $\begin{cases} x^2 - y^2 = 16 \\ 9x^2 - 25y^2 = 0 \end{cases}$

(c) $\begin{cases} x^2 - y^2 = 11 \\ 2x^2 - 5y^2 = 7 \end{cases}$

(j) $\begin{cases} x^2 + xy + y^2 = 4 \\ x^2 + 2y^2 = 12 \end{cases}$

(d) $\begin{cases} x^2 - xy = 3 \\ x^2 + y^2 = 5 \end{cases}$

(e) $\begin{cases} x^2 - y^2 = 13 \\ 2x^2 + y^2 = -1 \end{cases}$

(f) $\begin{cases} 4x^2 - y^2 = 4 \\ 4x^2 + y^2 = 4 \end{cases}$

(g) $\begin{cases} 2x^2 - 3xy = 2 \\ 2x^2 - 5xy - 3y^2 = 0 \end{cases}$

(k) $\begin{cases} 2x^2 + 2xy + y^2 - 5x + 6 = 0 \\ y^2 + 2xy + 5x - 6 = 0 \end{cases}$

(ℓ) $\begin{cases} 2x^2 + 3xy - y^2 = 4 \\ 2x^2 + 3xy - 8 = 0 \end{cases}$

(m) $\begin{cases} x^2 + xy + y^2 = 36 \\ xy = 0 \end{cases}$

(n) $\begin{cases} x^2 - 2xy = -1 \\ y^2 = xy \end{cases}$

7. Find the solution set of each of the following systems:

(a) $\begin{cases} x^2 + y^2 \leq 9 \\ y^2 \geq x \end{cases}$

(b) $\begin{cases} x^2 - y^2 \leq 9 \\ x^2 + 2y^2 \geq 4 \end{cases}$

(c) $\begin{cases} x^2 + y^2 \geq 25 \\ x^2 + y^2 \leq 25 \end{cases}$

(d) $\begin{cases} x^2 + y^2 < 16 \\ x + 2y - 2 = 0 \end{cases}$

(e) $\begin{cases} y < x^2 + 2x - 3 \\ 2x - y - 1 = 0 \end{cases}$

(f) $\begin{cases} x^2 + 16y^2 \geq 16 \\ x^2 + 4y^2 \leq 16 \end{cases}$

(h) $\begin{cases} x^2 + y^2 \geq 9 \\ 9x^2 + 16y^2 \leq 144 \\ x \geq 0 \end{cases}$

(i) $\begin{cases} y^2 \leq 4x \\ x^2 \geq 4y \\ x - y = 0 \end{cases}$

(j) $\begin{cases} x^2 + y^2 \leq 25 \\ \frac{x^2}{49} + \frac{y^2}{9} \geq 1 \end{cases}$

(k) $\begin{cases} y \leq x^2 - 6x + 9 \\ y \geq x^2 - 6x + 8 \end{cases}$

(ℓ) $\begin{cases} x^2 > 9 \\ 9 > y^2 \end{cases}$

(m) $\begin{cases} (x - 2)^2 + (y - 1)^2 \leq 4 \\ y \leq (x-2)^2 \end{cases}$

(g) $\begin{cases} x^2 + y^2 < 4 \\ x^2 - y^2 + 4 \geq 0 \end{cases}$ (n) $\begin{cases} y > 4 - x^2 \\ \frac{x^2}{16} + \frac{y^2}{4} < 1 \end{cases}$

7-6 Supplementary Exercises

1. Find the solution set of the following systems:

(a) $\begin{cases} x - 2y = 6 \\ x + y = 3 \end{cases}$ (e) $\begin{cases} \frac{x + y}{6} - \frac{x}{4} = \frac{1}{2} \\ \frac{2x - y}{6} - \frac{x}{4} = \frac{1}{2} \end{cases}$

(b) $\begin{cases} 5x + 8y = 7 \\ 2x + 7y = 1 \end{cases}$ (f) $\begin{cases} 2x - ay = b \\ 3x + 3y = d \end{cases}$

(c) $\begin{cases} 3x = 7 - y \\ 2x + 5y = 13 \end{cases}$ (g) $\begin{cases} x + ry = 2v \\ sx - y = 3w \end{cases}$

(d) $\begin{cases} 6x + 5 = y \\ \frac{1}{4}y = 4x \end{cases}$ (h) $\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$

2. Find the solution set of each of the following systems:

(a) $\begin{cases} x \geq 0 \\ y \geq 0 \end{cases}$ (d) $\begin{cases} x > 0 \\ y > 0 \\ y < -3x + 24 \\ y < -x + 10 \end{cases}$

(b) $\begin{cases} x < 2 \\ y > -3 \\ y - x \leq 0 \end{cases}$ (e) $\begin{cases} y < 2x^2 \\ \frac{x^2}{9} + \frac{y^2}{4} \leq 1 \\ y > 0 \end{cases}$

(c) $\begin{cases} 2x + 3y > 6 \\ y < x + 2 \\ x + y < 3 \end{cases}$ (f) $\begin{cases} x \geq -2 \\ x \leq 2 \\ y \geq -3 \\ y \leq 3 \end{cases}$

(g) $\begin{cases} x \geq 0 \\ y \geq 0 \\ 2x + 3y \geq 6 \end{cases}$

(h) $\begin{cases} 2x + 5y \geq 10 \\ 3x - 6y \geq 6 \\ 4x + 7y \leq 28 \\ 3x - 7y \leq 21 \end{cases}$

(i) $\begin{cases} 3x - 5y - 30 = 0 \\ 2x + 7y + 11 = 0 \\ 4x - y - 11 = 0 \end{cases}$

(j) $\begin{cases} xy = 12 \\ x + 2y = 0 \end{cases}$

(k) $\begin{cases} x^2 + y^2 = 1 \\ 4x^2 + 9y^2 = 36 \end{cases}$

(ℓ) $\begin{cases} 5x^2 + 4y^2 = 30 \\ 4x^2 + 5y^2 = 30 \end{cases}$

(m) $\begin{cases} x^2 + y^2 + 4x + 6y = 40 \\ x = 10 + y \end{cases}$

(n) $\begin{cases} 8x^2 - 3y^2 = 5 \\ 7x^2 - 3xy = 10 \end{cases}$

(o) $\begin{cases} x^2 + y^2 = 25 \\ 4y = x^2 - 20 \end{cases}$

(p) $\begin{cases} x^2 - 3xy = -4 \\ x^2 + 9y^2 = 20 \end{cases}$

(q) $\begin{cases} 2xy - y^2 = -24 \\ 2x^2 + xy = -2 \end{cases}$

(r) $\begin{cases} x^2 + y^2 = 36 \\ x^2 = y + 6 \end{cases}$

(s) $\begin{cases} 9x^2 + y^2 = 29 \\ 27x^2 - 2y^2 + 38 = 0 \end{cases}$

(t) $\begin{cases} x + y = 5 \\ \frac{1}{x} + \frac{1}{y} = \frac{5}{4} \end{cases}$

(u) $\begin{cases} x^2 = y \\ x + 1 = 0 \end{cases}$

(v) $\begin{cases} x^2 + 4y^2 = 144 \\ 4x^2 + y^2 = 144 \end{cases}$

(w) $\begin{cases} x^2 + 3y^2 = 3 \\ \frac{3}{2}x - y = 0 \end{cases}$

(x) $\begin{cases} x^2 + y^2 = 17 \\ 2y = 4x - 12 \end{cases}$

(y) $\begin{cases} x^2 - xy + y^2 = 12 \\ 3x = y + 10 \end{cases}$

*3. Find $\underline{m}$ in terms of $\underline{k}$ such that the line whose equation is $y = mx + k$ shall be tangent to the conic whose equation is $y = x^2$.

Chapter 8

SYSTEMS OF FIRST DEGREE EQUATIONS IN THREE VARIABLES

8-1. A Three Dimensional Coordinate System.

In Chapter 2 we learned that a one-to-one correspondence can be established between ordered pairs of real numbers (x,y) and points in a plane. In this chapter we shall deal with triples of numbers (x,y,z) and view them sometimes as constituting solution sets of equations in three variables, and sometimes as representing points in three-dimensional space. Thus we will wish, at the outset, to set up a one-to-one correspondence between ordered triples of real numbers (x,y,z) and the points of three-dimensional space. We use a method similar to the one we used in two dimensions.

Take three mutually perpendicular lines and label these lines the x-axis, the y-axis, and the z-axis respectively. These lines can be chosen, and labelled, in any manner whatsoever. For the sake of uniformity, and because the choice is a common one, let the x- and y-axes be in a horizontal plane and the z-axis perpendicular to this plane. The point of intersection of the axes is O, the origin. We assign number scales to the axes, as we did with coordinate systems in one and two dimensions, in such a way that the zero of each of the axes coincides at the origin. The positive direction $\overrightarrow{OX}$ extends forward, toward the observer; the positive direction $\overrightarrow{OY}$ extends to the right; and the positive direction $\overrightarrow{OZ}$ extends upward. A plane determined by any two

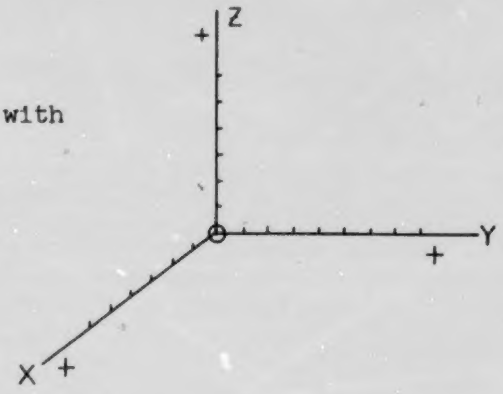


Figure 8-1a

of the axes is called a *coordinate plane*. There are three such planes, the XY-plane, the XZ-plane, and the YZ-plane.

Through any point P in space draw three planes which are respectively perpendicular to the three coordinate axes. The numbers attached to the points in which these planes intersect the x-, y-, and z-axes are called the x-coordinate, the y-coordinate, and the z-coordinate of the point P respectively. These planes and the three coordinate planes form a box-like figure (called a rectangular parallelepiped). We can then find the triple of coordinates of any given point in space; and, conversely, we can locate a point in space when any ordered triple of real numbers is given. *This one-to-one correspondence between points in space and the ordered triples of real numbers* (x, y, z) *is called a three-dimensional coordinate system.*

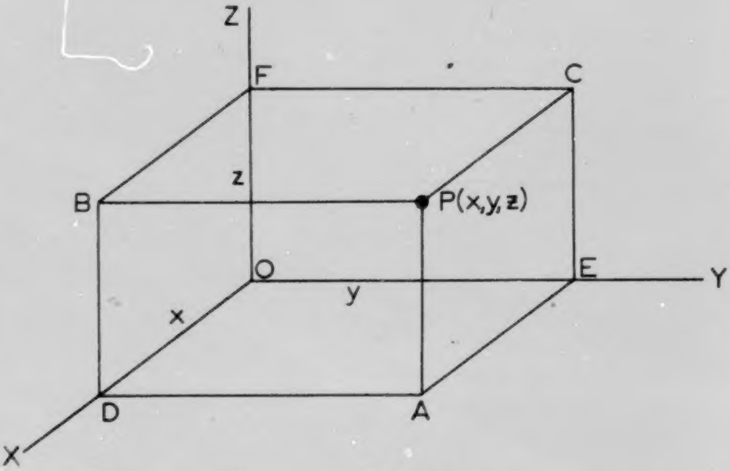


Figure 8-1b

Example: Plot the point (5,-2,4).

Solution: Begin at the origin and proceed 5 units in the direction of the positive x-axis, 2 units in the direction of the negative y-axis, and 4 units in the direction of the positive z-axis. The point located is the required point.

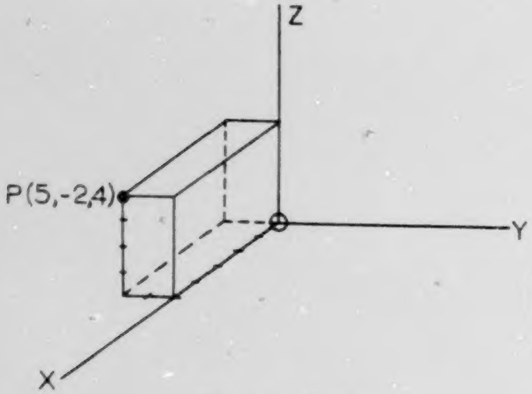


Figure 8-1c

Exercises 8-1.

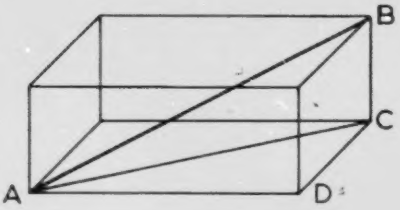
Plot the following points:

1. (0, -1, 3)
2. (-2, 0, 4)
3. (3, 2, 4)
4. (2, -1, -3)
5. (-4, -2, -7)
6. (0,2,0)
7. (1, -1, 0)
8. (2, -3, 4)
9. (3, 2, -4)
10. (2, 0, -3)
11. Where do all points lie for which $x = 0$; for which $x = 2$; for which $x = -3$?
12. Where do all points lie for which $y = 0$; for which $y = 3$?
13. Where do all the points lie for which $z = 2$; for which $z = -2$?

*14. Where do all points lie for which $x + y = 4$?

8-2. Distance Formula in Three Dimensions.

Development of a formula for the distance between two points in space is closely related to the problem of finding the length of the diagonal of a rectangular parallelepiped. Let us review the latter problem first. By virtue of the Pythagorean relation we have



$$d^2(A,C) = d^2(A,D) + d^2(D,C)$$

$$d^2(A,B) = d^2(A,C) + d^2(C,B)$$

Substituting for $d^2(A,C)$ we have

$$d^2(A,B) = d^2(A,D) + d^2(D,C) + d^2(C,B)$$

Figure 8-2a

$$(8\text{-}2a) \qquad d(A,B) = \sqrt{d^2(A,D) + d^2(D,C) + d^2(C,B)}$$

Thus, the diagonal of a rectangular parallelepiped equals the square root of the sum of the squares of its dimensions.

Consider now the distance between the points $A(1,2,4)$ and $B(3,5,6)$. These points are opposite vertices of a parallelepiped as indicated in Figure 8-2b. The distance between them, AB, may be obtained by applying formula

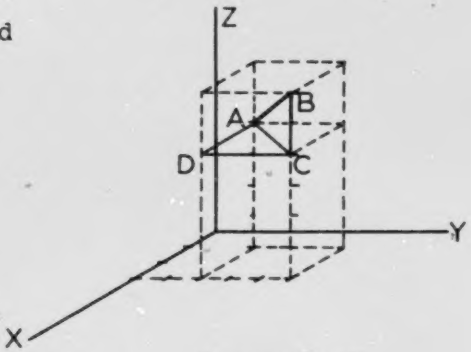


Figure 8-2b

$$(8\text{-}2a) \qquad d(A,B) = \sqrt{d^2(A,D) + d^2(D,C) + d^2(C,B)}.$$

From Figure 8-2b we see that

$$d(A,D) = 3 - 1 = 2$$
$$d(D,C) = 5 - 2 = 3$$
$$d(C,B) = 6 - 4 = 2$$
$$d(A,B) = \sqrt{4 + 9 + 4} = \sqrt{17}.$$

Using the same method, we now derive a formula for the distance between any two points in space, $P_1(x_1,y_1,z_1)$ and $P_2(x_2,y_2,z_2)$.

From (8-2a) we have

$$d(P_1,P_2) = \sqrt{d^2(P_1,Q) + d^2(Q,R) + d^2(R,P_2)} \quad \text{(See Fig.8-2c.)}$$

But $d(P_1,Q) = |x_2 - x_1|$

$d(Q,R) = |y_2 - y_1|$

$d(R,P_2) = |z_2 - z_1|$

$$(8\text{-}2b) \qquad \therefore \quad d(P_1,P_2) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

This is the formula for the distance between two points in three dimensions. The formula is correct no matter where P_1 and P_2 lie in space.

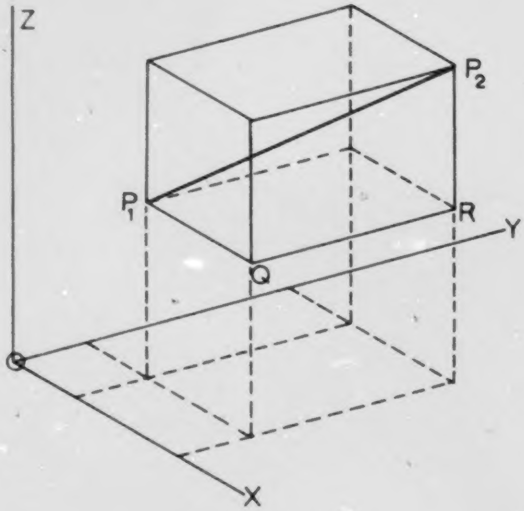


Figure 8-2c

Exercises 8-2.

Find the distance between the following pairs of points:

1. (6, 7, 1), (2, 3, 1)
2. (4, -1, -5), (7, 3, 7)
3. (0, 4, 5), (-6, 2, 8)
4. (3, 0, 7), (-1, 3, 7)
5. (4, -1, 3), (12, 7, -1)
6. (-4, 2, -7), (8, 18, 14)

7. (0, 1, 0), (-1, -1, -2)
8. (-3, 4, -8), (-8, -6, -6)
9. (3, 4, 5), (8, 4, 1)
10. (1, 2, 3), (0, 0, 0)

8-3. An Equation of a Plane.

From plane geometry we know that the set of points in a plane, at equal distances from two given points, is a line. Similarly, in space, the set of points at equal distances from two given points is a plane. We use this property to derive the equation of a plane. Since it was proved in geometry that this property characterizes a plane, the equation we derive will represent a plane with all the properties of the plane studied in geometry.

Example 1: Determine the equation of the plane whose points are equidistant from $A(1, 2, 3)$ and $B(2, 5, 4)$.

Solution: If $P(x, y, z)$ is any point in the plane, we know that

$$d(P,A) = d(P,B).$$

Using Formula (8-2b), we have

$$\sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2} = \sqrt{(x-2)^2 + (y-5)^2 + (z-4)^2}\,.$$

From this we have

$$x^2 - 2x + 1 + y^2 - 4y + 4 + z^2 - 6z + 9 = x^2 - 4x + 4 + y^2 - 10y + 25 + z^2 - 8z + 16$$

which reduces to

$$2x + 6y + 2z = 31. \tag{8-3a}$$

Thus the equation of this plane is of first degree in 3 variables.

Using this same method we prove that the equation of every plane is an equation of first degree in 3 variables. Instead of two special points, A and B, we use $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ to represent any two distinct points in space. Then we have

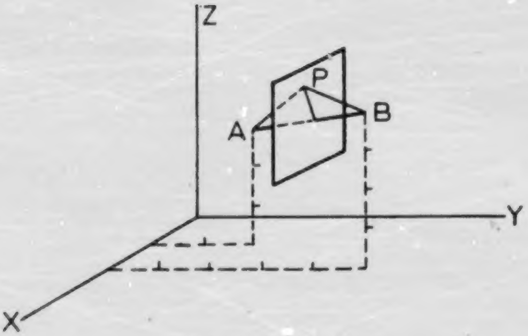


Figure 8-3a

$$d(P_1,P) = d(P_2,P)$$

$$\sqrt{(x_1-x)^2 + (y_1-y)^2 + (z_1-z)^2} = \sqrt{(x_2-x)^2 + (y_2-y)^2 + (z_2-z)^2}$$

$$x_1^2 - 2x_1x + x^2 + y_1^2 - 2y_1y + y^2 + z_1^2 - 2z_1z + z^2$$

$$= x_2^2 - 2x_2x + x^2 + y_2^2 - 2y_2y + y^2 + z_2^2 - 2z_2z + z^2$$

$$(8\text{-}3b) \quad 2(x_2 - x_1)x + 2(y_2 - y_1)y + 2(z_2 - z_1)z - [(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + (z_2^2 - z_1^2)] = 0.$$

Since $d(P_1,P)$ and $d(P_2,P)$ are positive numbers, this argument can be reversed. Therefore we know that a point $P(x, y, z)$ whose coordinates satisfy equation (8-3b) is equidistant from P_1 and P_2.

Equation (8-3b) is an equation of first degree provided the coefficients of x, y, and z are not all zero. Let us denote these coefficients by

$$A = 2(x_2 - x_1), \quad B = 2(y_2 - y_1), \quad C = 2(z_2 - z_1).$$

These will all be zero only if $x_2 = x_1$, $y_2 = y_1$, and $z_2 = z_1$, i.e., points P_1 and P_2 coincide. But P_1 and P_2 are distinct. Therefore we have proved that every plane in three dimensions can be represented by an equation of the form

$$Ax + By + Cz + D = 0$$

where

$$D = -[(x_2^2 - x_1^2) + (y_2^2 - y_1^2) + (z_2^2 - z_1^2)]$$

and A, B, C are real constants, not all zero. The converse theorem can also be proved, i.e., that every equation of this form represents a plane. The proof of this converse is given below.

Proof: Let $P(x, y, z)$ be any point on the plane that is the set of points equidistant from $O(0, 0, 0)$ and $Q(kA, kB, kC)$ where

$$k = \frac{-2D}{A^2 + B^2 + C^2}$$

Then $\quad PO = PQ$

$$x^2 + y^2 + z^2 = (x - kA)^2 + (y - kB)^2 + (z - kC)^2$$

$$0 = -2kAx + k^2A^2 - 2kBy + k^2B^2 - 2kCz + k^2C^2$$

$$2k(Ax + By + Cz) = k^2(A^2 + B^2 + C^2)$$

$$Ax + By + Cz = \frac{k}{2}(A^2 + B^2 + C^2) .$$

$$\text{Put } k = \frac{-2D}{A^2 + B^2 + C^2}$$

The equation becomes

$$Ax + By + Cz + D = 0$$

This argument is reversible. This means that any point P whose coordinates satisfy $Ax + By + Cz + D = 0$ is equidistant from the two points O and Q. Hence $Ax + By + Cz + D = 0$ is, by definition, the equation of a plane.

Note: If $D = 0$, it follows that $k = 0$. The two points coincide, and no plane is determined. The case where $D = 0$ is treated in Problem 3, Exercise 8-3.

Exercises 8-3.

1. Use the method of Example 1 to find the equation of the plane whose points are equidistant from each of the following pairs of points:
   (a) (-1, 3, 2), (4, -2, -2);
   (b) (-1, -3, -2), (-2, 0, 4);
   (c) (5, -1, 2), (-5, 1, -2);
   (d) (2, 4, -5), (0, 2, 3);
   (e) (-2, 0, 6), (1, 4, 3);
   (f) (-1, 2, -3), (1, -2, 3).
2. In each of the following, find the equation of the plane that is the set of points equidistant from the given points, and sketch the graph.
   (a) (4, 0, 0), (-2, 0, 0)
   (b) (0, 3, 0), (0, -1, 0)
   (c) (0, 0, 0), (4, 2, 0)
   (d) (0, 0, 0), (0, 5, 3)

*3. Prove that the equation

$$ax + by + cz = 0$$

where not all the constants a, b, c are zero, represents the set of points equidistant from the symmetric points (a, b, c) and (-a, -b, -c).

8-4. The Solution Set of an Equation in Three Variables.

We shall examine several first degree equations in three variables, both graphically and algebraically, to gain familiarity with this representation of a plane.

Definition 8-4a. The solution set of an equation in three variables is the set of real number triples (x, y, z) that satisfy the equation.

Example 1: Find some of the elements of the solution set of the equation

(8-4a) $$x + 2y + z = 5.$$

Solution: We may tabulate elements of the solution set of this equation by assigning values to x and y, and computing the corresponding values of z. In this way we may find as many number triples of the solution set as we wish.

In the first lines of the tabulation given below, we give the assigned values of x and y; in the third line we give the computed value of z.

x	0	1	-1	1	2	0
y	0	1	1	-1	0	2
z	5	2	4	6	3	1

x arbitrary

y arbitrary

$z = 5 - x - 2y$

Example 2: By considering sets of points in the solution set of

$$x + y = 4,$$

sketch the graph of the equation.

Solution: Viewed as an equation in three variables, this equation has the form

(8-4b) $$x + y + 0 \cdot z = 4.$$

Since the coefficient of z in this equation is zero, we are no longer free to assign values to x and y at random. For example, if $x = 1$, we must assign the value 3 to y. On the other hand, when $x = 1$ and $y = 3$, we are free to assign any value whatsoever to z. We know from the definition of the coordinates of a point $P(x, y, z)$ (see Figure 8-1b) that all the points for which $x = 1$ and $y = 3$ lie on the perpendicular to the XY plane through the point $(1, 3, 0)$. Since all these points $(1, 3, z)$ correspond to number triples in the solution set of equation (8-4b) no matter what value z has, we see that this perpendicular line lies in the plane $x + y + 0 \cdot z = 4$

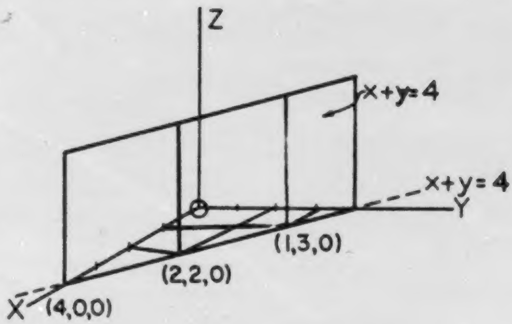


Figure 8-4a

(Figure 8-4a). Similarly all points (2, 2, z), (3, 1, z), (4, 0, z) lie in the plane. Continuing in this fashion, we see that the plane contains all the perpendiculars to the XY-plane that intersect the XY-plane in the line $x + y = 4$. Since all these lines lie in a plane perpendicular to the XY-plane, we see that the equation $x + y = 4$ represents a plane perpendicular to the XY-plane. Its line of intersection with the XY-plane has the equation $x + y = 4$ $(z = 0)$.

Example 3: By considering subsets of the solution set of the equation $x = 3$, sketch a graph of the equation.

Solution: Viewed as an equation in three variables, this equation has the form

$$x + 0 \cdot y + 0 \cdot z = 3.$$

Here x must be assigned the value 3, but y and z may assume any values. We see then that this plane is the set of points at the directed distance, +3, from the YZ-plane. It is therefore parallel to the YZ-plane.

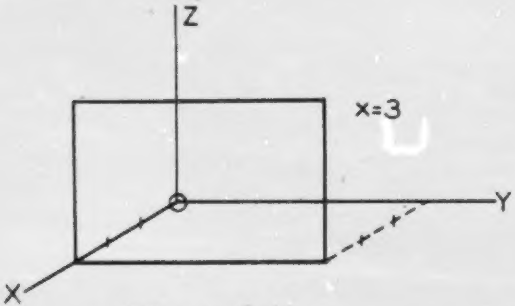


Figure 8-4b

Exercises 8-4.

1. Sketch the graphs of each of the equations

   (a) $x - 2y = 5$ (d) $y - 2z = 0$

   (b) $x - 2y = 0$ (e) $2x - z = 0$

   (c) $y + 2z = 8$

2. Four points on the graph of the plane

   $$2x + y = 6$$

   are seen to be $A(3, 0, 0)$, $B(1, 4, 0)$, $C(2, 2, 0)$, $D(0, 6, 0)$. Give three other points on the graph with the same x and y values as A; as B; as C; as D. Sketch the graph.

3. Sketch the graph of $z = -2$; of $x = 5$; of $y = 3$.

8-5. The Graph of a First Degree Equation in Three Variables.

If either one or two of the coefficients in the equation

$$Ax + By + Cz + D = 0$$

are zero, Section 8-4 gives us a method of graphing the equation. If all the coefficients are different from zero, we proceed in a similar fashion.

Consider, for example, the graph of the equation

$$x + 2y + z = 5. \tag{8-5a}$$

Recall from Chapter 6 that an easy way to plot the graph of a linear equation is to find the intercepts of the line. Similarly in three dimensions the graph of a plane is easy to sketch if we begin by finding the intersection of the plane with the coordinate planes. These intersections with the coordinate planes are called traces. If we want the intersection of plane (8-5a) with the XY-plane we must put $z = 0$ in the equation

$$x + 2y + z = 5.$$

The resulting equation is

$$x + 2y = 5.$$

This is the equation of a straight line in the XY-plane, and this straight line is called the trace of

$$x + 2y + z = 5$$

in the XY-plane.
Similarly the XZ-trace is

$$x + z = 5,$$

and the YZ-trace is

$$2y + z = 5.$$

The graph of these lines in the coordinate planes makes the position of the plane

$$x + 2y + z = 5$$

clear.

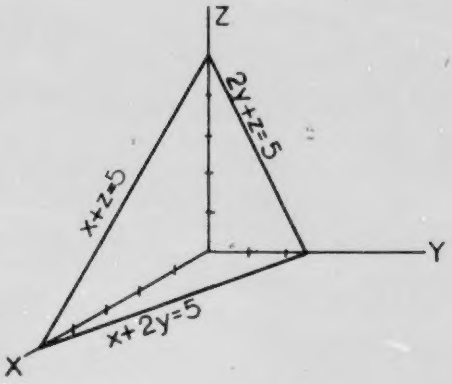


Figure 8-5

Exercises 8-5.

1. Sketch the graph of each of the following equations.
   (a) $x - 2y + z = 5$
   (b) $x + z = 5$
   (c) $x - 2y - z = 5$
   (d) $x + 2y + z = 5$
   (e) $4x - 2y + z = 0$
   (f) $5x + 4y = 20$
   (g) $3x - 2y + \frac{5}{3} z = 0$
   (h) $- \frac{x}{5} + \frac{y}{3} + \frac{z}{6} = 1$
   (i) $x - 2y - z = 0$
   (j) $2 \frac{1}{2} x - 3 \frac{1}{3} y = 0$
2. On the same set of axes sketch the graphs of the following pairs of equations, indicating the graph of the intersection set.
   (a) $x + 2y + z = 5$, $x = 1$
   (b) $x - 2y + z = 5$, $z = 2$

(c) $5x + 4y = 20$
$3x - 4y = 0$

(d) $5x + 4y = 20$
$-9x + 6y - 5z = 0$

(e) $x - 2y + z = 5$
$2x - 4y + 2z = 10$

8-6. The Solution Set of a System of First Degree Equations in Three Variables. Definitions.

Definition 8-6a. A system of first degree equations in three variables consists of two or more equations in three variables. In this book we will consider only systems that involve either two or three equations.

Definition 8-6b. The solution set of a system of first degree equations in three variables is the set of all number triples that satisfy all equations of the system. (It is the intersection of the solution sets of the equations of the system.)

Definition 8-6c. Two systems are equivalent if their solution sets are the same.

*8-7. The Solution Set of a System of Two First Degree Equations in Three Variables. Graphic Approach. (See Appendix.)

8-8. Algebraic Representation of the Line of Intersection of Two Intersecting Planes.

In this section we study the intersection of a pair of planes

$$x + 2y - z - 5 = 0 \tag{8-8a}$$

$$x + y + z - 2 = 0. \tag{8-8b}$$

Our procedure is to obtain the equations of three planes which pass through the line of intersection of the given planes and which give particularly useful representations of that line. We construct three different linear combinations of the expressions

$$(x + 2y - z - 5)$$

and $$(x + y + z - 2),$$

and find three components of equivalent systems each of which has the coefficient of at least one variable equal to zero.

A component of an equivalent system can be written

$$a(x + 2y - z - 5) + b(x + y + z - 2) = 0.$$

(1) We eliminate x by choosing $a = 1, b = -1$.

$$(x + 2y - z - 5) - (x + v + z - 2) = 0$$

(8-8c) $$y - 2z - 3 = 0$$

(2) We eliminate y by choosing $a = 1, b = -2$.

$$(x + 2y - z - 5) - 2(x + y + z - 2) = 0$$

(8-8d) $$-x - 3z - 1 = 0$$

(3) We eliminate z by choosing $a = 1, b = 1$.

$$(x + 2y - z - 5) + (x + y + z - 2) = 0$$

(8-8e) $$2x + 3y - 7 = 0$$

We now have three distinct new equations (8-8c), (8-8d), (8-8e), any two of which may be chosen to represent the line of intersection of the given planes.

If we represent this line by the planes

(8- 8c) $$y - 2z - 3 = 0$$

and

(8-8d) $$-x - 3z - 1 = 0$$

we can express x and y in terms of z:

(8-8f) (i) $$\begin{cases} x = -3z - 1, \\ y = 2z + 3. \end{cases}$$

This is an especially convenient form for determining particular points on the line of intersection of the two given planes. It enables us easily to write down as many number triples in the solution set as we wish. We see that we may assign values to z at random, and obtain corresponding values of x and y. Thus the solution set contains infinitely many number triples. This is what we should have expected, since the intersection of these two planes is a line.

Example 1. Write 4 members of the solution set of the above system (1).

Solution: Using the first representation given above, (8-8f), assign arbitrary values to z, and compute the corresponding values of x and y.

x	-1	-4	2	-7
y	3	5	1	7
z	0	1	-1	2

$$\begin{cases} x = -3z - 1 \\ y = 2z + 3 \\ z \text{ arbitrary} \end{cases}$$

If we use (8-8d) and (8-8e), we can express y and z in terms of x;

$$\text{(8-8g)} \qquad \begin{aligned} y &= -\tfrac{1}{3}(2x - 7), \\ z &= -\tfrac{1}{3}(x + 1). \end{aligned}$$

Using this representation of the line of intersection of the two given planes, check the number of triples obtained above by assigning the tabulated values of x, and computing the other values.

x	-1	-4	2	-7
y				
z				

$$\begin{cases} x \text{ arbitrary} \\ y = -\tfrac{1}{3}(2x - 7) \\ z = -\tfrac{1}{3}(x + 1) \end{cases}$$

Using (8-8c) and (8-8e) we can express x and z in terms of y:

$$\text{(8-8h)} \qquad \begin{aligned} x &= -\tfrac{1}{2}(3y - 7), \\ z &= \tfrac{1}{2}(y - 3). \end{aligned}$$

Using this representation, check again the number triples obtained from (8-8f) by assigning the tabulated values of y, and computing the corresponding values of x and z.

x				
y	3	5	1	7
z				

$$\begin{cases} x = -\frac{1}{2}(3y - 7) \\ y \ \text{arbitrary} \\ z = \frac{1}{2}(y - 3) \end{cases}$$

Example 2: Find four number triples in the solution set of the system

$$\begin{aligned} 2x - y + 2z &= 6, \\ z &= 2. \end{aligned}$$

How can we describe the whole solution set algebraically?

Solution: In this example every number triple in the solution set has $z = 2$. By substituting this value in the first equation we have

$$2x - y = 2$$

$$y = 2x - 2$$

$$\text{or} \qquad x = \frac{1}{2}(y + 2).$$

Thus four number triples in the solution set can be written by assigning arbitrary values to x, and computing the values of y:

x	0	1	-1	2
y	-2	0	-4	2
z	2	2	2	2

$$\begin{cases} x \ \text{arbitrary} \\ y = 2x - 2 \\ z = 2 \end{cases}$$

or by assigning arbitrary values to y, and computing the values of x:

x				
y	-2	0	-4	2
z				

$$\begin{cases} x = \frac{1}{2}(y + 2) \\ y \ \text{arbitrary} \\ z = 2 \end{cases}$$

The complete description of the solution set is given either

$$\text{as} \quad \begin{cases} x \ \text{arbitrary} \\ y = 2x - 2 \\ z = 2 \end{cases} \quad \text{or as} \quad \begin{cases} x = \frac{1}{2}(y + 2) \\ y \ \text{arbitrary} \\ z = 2. \end{cases}$$

In this case, z may not be chosen arbitrarily.

Exercises 8-8.

In each of the problems given below, if the planes intersect in a line, express two of the variables of the solution set in terms of the third variable, and tabulate a subset of the solution set consisting of four number triples.

1. $x - 3y - z = 11$
   $x - 5y + z = 1$
2. $x + 2y - z = 8$
   $x + y + z = 0$
3. $x - z + y = 5$
   $x + 2y = -z$
4. $2x + 4y - 7 = 5z$
   $4x + 8y - 14 = 5z$
5. $-2x + y + 3z = 0$
   $-4x + 2y + 6z = 0$
6. $2x + 6z - 18y = 6$
   $x - 3z - y = -3$
7. $3x - 4y + 2z = 6$
   $6x - 8y + 4z = 14$
8. $-5x + 4y + 8z = 0$
   $-3x + 5y + 15z = 0$
9. $6z - 7y + 4x = 13$
   $5x + 6y - z = 7$
10. $-10x + 4y - 5z = 20$
    $2x - \frac{4}{5}y + z = 4$

8-9. The Solution Set of a System of Three First Degree Equations in Three Variables.

We now consider the solution set of three first degree equations in three variables. A simple example will introduce us to the problem.

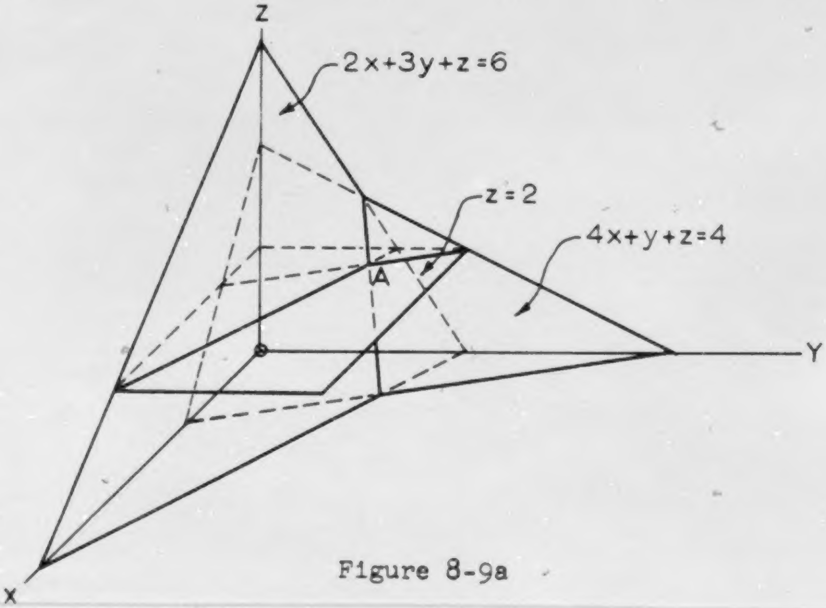


Figure 8-9a

$$
\begin{aligned}
2x + 3y + z &= 6,\\
4x + y + z &= 4,\\
z &= 2.
\end{aligned}
$$

Figure 8-9a suggests the graphic solution in which A is the single point of intersection of the three planes. Algebraically, we may use the value of z given by the third equation; substitute it in the first two equations, and then solve for x and y:

$$
\begin{aligned}
&2x + 3y = 4,\\
&4x + y = 2;\\
&x = \frac{1}{5}\ ;\ y = \frac{6}{5}\ ;\ z = 2.
\end{aligned}
$$

The point of intersection of the three planes is $(\frac{1}{5}, \frac{6}{5}, 2)$.

Usually the graphic representation of the three planes represented by three first degree equations in three variables will be too complicated to draw. But it is helpful to keep in mind the geometric meaning of the equations when we consider the types of solution sets that we may expect. These correspond to the types of intersections that are possible for three planes in space. The method of solution will be the same in all cases. It is illustrated by the following examples. In each case the problem is to find the solution set.

Example 1:

$$
\begin{aligned}
x + 2y - 3z &= 9,\\
2x - y + 2z &= -8,\\
-x + 3y - 4z &= 15.
\end{aligned}
$$

Step 1: Eliminate x from the second and third equations by adding appropriate multiples of the first equation. We now have the equivalent system

$$
\begin{aligned}
x + 2y - 3z &= 9,\\
0 - 5y + 8z &= -26,\\
0 + 5y - 7z &= 24.
\end{aligned}
$$

Step 2: Eliminate y from the third equation by adding an appropriate multiple of the second equation obtaining the equivalent system

$$\begin{aligned} x + 2y - 3z &= 9, \\ 0 - 5y + 8z &= -26 \\ 0 + 0 + z &= -2. \end{aligned}$$

Step 3: Substitute $z = -2$ in the second equation obtaining

$$\begin{aligned} -5y &= -26 + 16 \\ y &= 2. \end{aligned}$$

Step 4: Substitute $z = -2$, and $y = 2$ in the first equation.

$$\begin{aligned} x + 4 + 6 &= 9 \\ x &= -1. \end{aligned}$$

Step 5: Check the solution.

$$\begin{aligned} -1 + 4 + 6 &= 9, \\ -2 - 2 - 4 &= -8, \\ 1 + 6 + 8 &= 15. \end{aligned}$$

We see that the solution is the number triple (-1, 2, -2). The planes intersect in a point. Figure 8-9b, page 433, shows three planes intersecting in a point. (Case 1.)

Example 2:

$$\begin{aligned} 2x - 3y + z - 3 &= 0, \\ x + 5y - z - 3 &= 0, \\ 5x + 12y - 2z - 12 &= 0. \end{aligned}$$

To simplify the arithmetic, we interchange the first and second equations, and proceed with the steps described in Example 1.

Step 1: Eliminate x from two equations, obtaining the equivalent system,

$$\begin{aligned} x + 5y - z - 3 &= 0, \\ 0 - 13y + 3z + 3 &= 0, \\ 0 - 13y + 3z + 3 &= 0. \end{aligned}$$

Step 2: Eliminate y from the third equation.

$$\begin{aligned} x + 5y - z - 3 &= 0, \\ 0 - 13y + 3z + 3 &= 0, \\ 0 + 0 + 0 + 0 &= 0. \end{aligned}$$

In this case, the third equation contributes no new information. If Step 2 gives the identity, $0 = 0$, one of the given equations is a linear combination of the other two. Here the left member of the third equation,

$$5x + 12y - 2z - 12$$

can be obtained as

$$(2x - 3y + z - 3) + 3(x + 5y - z - 3).$$

Therefore, we know, by an argument similar to that given in the discussion of equivalent systems in Chapter 7, that the graph of the third equation must pass through the line of intersection of the planes

$$2x - 3y + z - 3 = 0$$
$$x + 5y - z - 3 = 0.$$

(This relationship will be studied further in Section 8-10.) Thus the complete solution of the given system is an infinite set of triples representing the points on the line of intersection of the given planes. We may use the method in Section 8-8 if we wish to determine the numbers of the solution set.

Eliminating x from the first two equations we have

$$(2x - 3y + z - 3) - 2(x + 5y - z - 3) = 0$$
$$-13y + 3z + 3 = 0.$$

Eliminating z from the first two equations we have

$$(2x - 3y + z - 3) + (x + 5y - z - 3) = 0$$
$$3x + 2y - 6 = 0.$$

Solving for x and z in terms of y:

$$x = \frac{1}{3}(-2y + 6),$$
$$z = \frac{1}{3}(13y - 3),$$
$$y \text{ is arbitrary.}$$

Figure 8-9b shows three planes intersecting in a straight line. (Case 2a.)

Example 3:

$$x + 5y - z - 3 = 0,$$
$$2x - 3y + z - 3 = 0,$$
$$2x + 10y - 2z - 6 = 0.$$

Here Step 1 yields

$$x + 5y - z - 3 = 0,$$
$$0 - 13y + 3z + 3 = 0,$$
$$0 + 0 + 0 + 0 = 0.$$

We see that the left-hand member of the third equation is twice the left-hand member of the first equation.

$$2x + 10y - 2z - 6 = 2(x + 5y - z - 3).$$

Therefore the first and third planes coincide. Again, the solution is completely described by the first two equations. It is the same line we found in Example 2. (See Case 2b, Figure 8-9b.)

Example 4:
$$x + 2y + z = 4,$$
$$x - 2y + z = 0,$$
$$x + z = 4.$$

For simplicity, move the third equation into the first row

$$x + z = 4,$$
$$x + 2y + z = 4,$$
$$x - 2y + z = 0.$$

Step 1: Eliminate x in the second and third equations.

$$x + 0 + z = 4,$$
$$0 + 2y + 0 = 0,$$
$$0 - 2y + 0 = -4.$$

Step 2: Eliminate y from the new third equation.

$$x + 0 + z = 4,$$
$$0 + 2y + 0 = 0,$$
$$0 + 0 + 0 = -4.$$

Since there are no triples for which $0 = -4$ there are no solutions. In this case one plane is parallel to the intersection of the other two. (See Case 4a, Figure 8-9b.)

Example 5:
$$x + y + 2z = 1,$$
$$x + y + 2z = 2,$$
$$x + y + 2z = 3.$$

By subtracting the first equation from the other two we find immediately

$$\begin{aligned} x + y + 2z &= 1, \\ 0 + 0 + 0 &= 1, \\ 0 + 0 + 0 &= 2. \end{aligned}$$

Again, we have no solution. The three planes are parallel. (See Case 4d, Figure 8-9b.)

Example 6:

$$\begin{aligned} x - y - 2z &= 1, \\ 2x - 2y - 4z &= 2, \\ -x + y + 2z &= -1. \end{aligned}$$

Step 1 gives the equivalent system

$$\begin{aligned} x - y - 2z &= 1, \\ 0 + 0 + 0 &= 0, \\ 0 + 0 + 0 &= 0. \end{aligned}$$

In this case, the three equations represent the same plane. (See Case 3, Figure 8-9b.)

Example 7:

$$\begin{aligned} \frac{3}{x} + \frac{2}{y} - \frac{1}{z} &= 3, \\ \frac{2}{x} + \frac{3}{y} + \frac{2}{z} &= 3, \\ \frac{4}{x} + \frac{1}{y} - \frac{3}{z} &= 4. \end{aligned}$$

These equations are linear in the variables $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$. We treat these reciprocals as the unknowns.

Retain the first equation, changing the order of the variables so that the computation that follows can be carried on more conveniently.

$$-\frac{1}{z} + \frac{2}{y} + \frac{3}{x} = 3$$

Step 1: Eliminate $\frac{1}{z}$ from the second and third equations. We have the equivalent system

$$\begin{aligned} -\frac{1}{z} + \frac{2}{y} + \frac{3}{x} &= 3, \\ 0 + \frac{7}{y} + \frac{8}{x} &= 9, \\ 0 + \frac{5}{y} + \frac{5}{x} &= 5 \quad \text{or} \quad 0 + \frac{1}{y} + \frac{1}{x} = 1. \end{aligned}$$

Step 2: Eliminate $\frac{1}{y}$ from the third equation

$$-\frac{1}{z} + \frac{2}{y} + \frac{3}{x} = 3,$$

$$0 + \frac{7}{y} + \frac{8}{x} = 9,$$

$$0 + 0 + \frac{1}{x} = 2. \qquad \therefore\ x = \frac{1}{2}\ .$$

Step 3: Substitute $\frac{1}{x} = 2$ in the second equation.

$$\frac{7}{y} + 16 = 9$$

$$\frac{7}{y} = -7$$

$$\frac{1}{y} = -1 \qquad \therefore\ y = -1.$$

Step 4: Substitute $\frac{1}{x} = 2$, $\frac{1}{y} = -1$ in the first equation, obtaining

$$-\frac{1}{z} - 2 + 6 = 3$$

$$\frac{1}{z} = 1. \qquad \therefore\ z = 1.$$

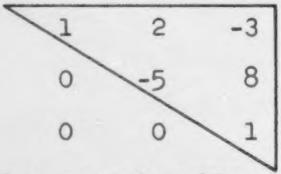
Step 5: Check the solution.

$$3 \cdot 2 + 2(-1) - 1 = 3,$$

$$2 \cdot 2 + 3(-1) + 2 = 3,$$

$$4 \cdot 2 + (-1) - 3 = 4.$$

Summary. The method described in this section is called triangulation because, in the case of a unique solution, the nonzero coefficients (represented by Step 2 in Example 1) lie in the form of a triangle:



This method provides a systematic procedure that enables us to recognize when the solution set is empty, when it contains a single triple, and when it contains infinitely many triples corresponding either to a line of points or to a plane of points. The method can be summarized as follows:

Step 1. After choosing a convenient first equation, eliminate one variable (say x) from the other two equations by adding appropriate multiples of the (chosen) first equation.

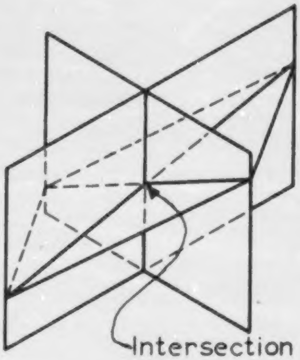
Step 2. In a similar way, work with the second and third equations which now contain only y and z. Multiplying by suitable numbers, eliminate a second variable (say y) from the third equation.

Steps 3 and 4. The third equation now gives a value of one variable (say z). Substitute this value in the second equation to obtain y. Substitute the values of y and z in the first equation to obtain x.

Step 5. Check the values of x, y, z found in Steps 1 - 4 in the given equations.

In Figure 8-9b we give sketches that illustrate the possible types of intersection of three planes in space.

1. The three planes intersect in a point. The solution set is a single number triple.



2. The three planes intersect in a line. The solution set is the infinite set of number triples corresponding to the points on the line.
   (a) The three planes have a line in common.

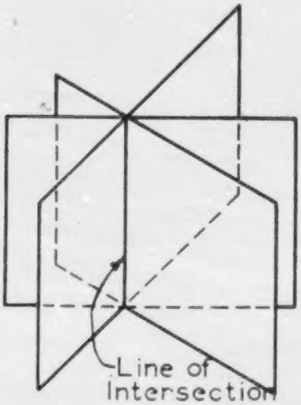
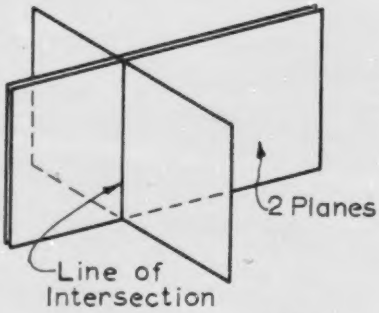
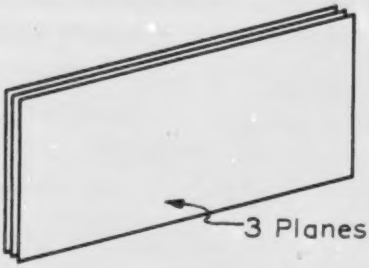


Figure 8-9b

2. (b) Two planes coincide and intersect the third plane in a line. The solution set is the same as in 2(a).


2 Planes
Line of Intersection

3. All three planes coincide. The solution set is the infinite set of number triples corresponding to the points in the plane.



4. The three planes do not have a common intersection. The solution set is empty. The system is inconsistent.
   (a) Two planes intersect; the third is parallel to their intersection.

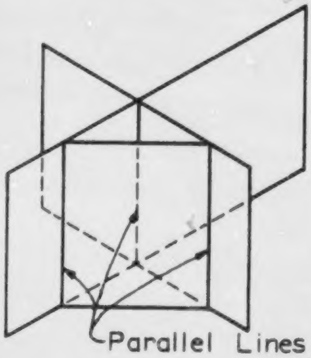
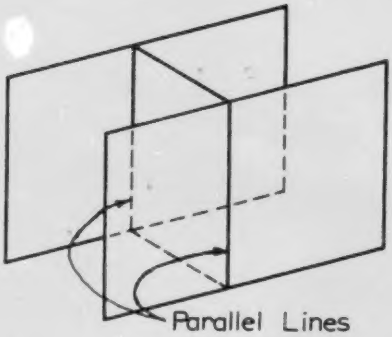
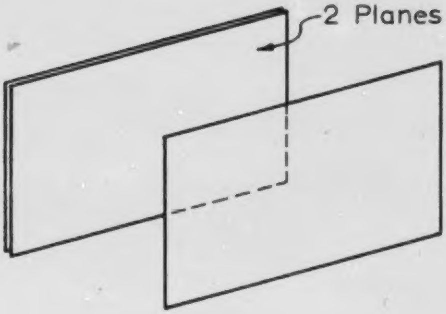

Parallel Lines

Figure 8-9b continued

4. (b) Two planes are parallel. The third plane intersects these two in parallel lines.


Parallel Lines

(c) Two planes coincide and are parallel to the third plane.



(d) The three planes are parallel.

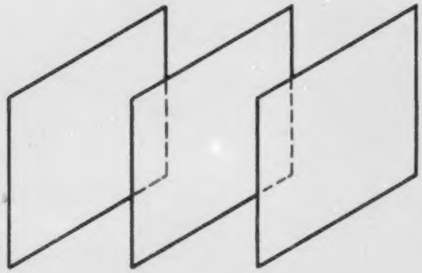


Figure 8-9b continued

Exercises 8-9.

In each of the following problems, determine whether the solution set is empty or whether its graph is a point, a line, or a plane. If the intersection is a point, give its coordinates.

1. $x + z = 8,$
   $x + y + 2z = 17,$
   $x + 2y + z = 16.$
2. $x + 2y - z = 5,$
   $x + y + 2z = 11,$
   $x + y + 3z = 14 .$
3. $x + 2y - z = -1,$
   $2x + 2y - 3z = -1,$
   $4x - y + 2z = 11.$
4. $x + y - 5z = 9,$
   $2x + 3y - 12z = 22,$
   $3x - 5y + z = -5.$
5. $x - 2y + 3z = 6,$
   $2x + y - 2z = -1,$
   $3x - 3y - z = 5.$
6. $2x + 4y + z = 0,$
   $x - y + 3z = 8,$
   $3x + y - 2z = -2.$
7. $x - 2y + z = 4,$
   $-3x + 6y - 3z = -12,$
   $2x - 4y + 2z = 8.$
8. $2x + 3y + 7z - 13 = 0,$
   $3x + 2y - 5z + 22 = 0,$
   $5x + 7y - 3z + 28 = 0.$
9. $4x - y + z = 6,$
   $3x + 2y - 4z = 2,$
   $7x + y - 3z = 5.$
10. $20x - 20y - 30z = 0,$
    $15x - 10y - 25z = 0,$
    $10x - 20y - 10z = 0.$
11. $\frac{10}{3x} + \frac{3}{y} + \frac{2}{z} = 2,$
    $\frac{10}{3x} + \frac{1}{y} - \frac{3}{z} = 0,$
    $\frac{4}{x} - \frac{18}{y} + \frac{3}{z} = -1.$
12. $\frac{5}{x} + \frac{12}{y} + \frac{10}{z} = 1,$
    $\frac{3}{x} - \frac{8}{y} - \frac{2}{z} = 1,$
    $-\frac{4}{x} - \frac{8}{y} - \frac{3}{z} = -5.$
13. $x + y + z = 2,$
    $2x + 2y + 2z = 5,$
    $x - y + z = 7.$
14. $3x - y - 2z - 2 = 0,$
    $2y - z + 1 = 0,$
    $3x - 5y - 3 = 0.$
15. $\frac{3}{x} - \frac{1}{y} = 7,$
    $\frac{3}{y} - \frac{1}{z} = 5,$
    $\frac{3}{z} - \frac{1}{x} = 0.$
16. $x + y + z = 3,$
    $3x + 3y + 3z = 9,$
    $x + y - z = 6.$

17. $\frac{1}{x} + \frac{2}{y} - \frac{1}{z} = 5,$

$\frac{2}{x} - \frac{1}{y} + \frac{1}{z} = 1,$

$\frac{1}{z} - \frac{3}{y} - \frac{1}{x} + 7 = 0.$

18. $x + 2y + z = 3,$

$2x - y + 3z = 7,$

$3x + y + 4z = 10.$

19. $3x + 5y + 2z = 0,$

$12x - 15y + 4z = 12,$

$6x - 25y - 8z = 8.$

20. $2x - y + 4z = 3,$

$3x + 2y - 2z = -1,$

$x - 4y + 10z = 7.$

21. $2x + y + z - 3 = 0,$

$x + 4y + 3z - 10 = 0,$

$x - 3y - 2z + 7 = 0.$

22. $x - 2y - 3z = 2,$

$x - 4y - 13z = 14,$

$3x - 5y - 4z = 0.$

*23. We consider buying three kinds of food. Food I has one unit of vitamin A, three units of vitamin B, and four units of vitamin C. Food II has two, three and five units, respectively. Food III has three units each of vitamin A and vitamin C, none of vitamin B. We need to have 11 units of vitamin A, 9 of vitamin B, and 20 of vitamin C.

(a) Have we enough information to determine uniquely the amounts of each of the foods we must get?

(b) Suppose Food I costs 60 cents and the others 10 cents per unit. Is there a solution for this problem if exactly one dollar is spent for these foods?

*24. The solution set of the following system contains only one triple. Determine which of the equations may be omitted without altering the solution set.

$$\begin{cases} x + y = 5 \\ -x + 3z = 2 \\ x + 2y + z = 1 \\ y + z = -4 \end{cases}$$

---

*8-10. Equivalent Systems of Equations in Three Variables.
(See Appendix.)

8-11. Miscellaneous Exercises.

1. A number may be written in the form $100h + 10t + u$, where h, t, and u represent respectively the hundreds, tens and units digits. If the sum of the digits of a certain number is 13, the sum of the units and tens digits is 10, and the number is increased by 99 if the digits are reversed, find the number.

2. Find the relation that must hold between the numbers a, b, c in order that the system

$$\begin{cases} 3x + 4y + 5z = a, \\ 4x + 5y + 6z = b, \\ 5x + 6y + 7z = c, \end{cases}$$

have a solution.

3. Find a three digit number such that the difference between each succeeding pair of digits is 1 and the sum of the digits is 15.

4. A man has three sums of money invested, one at 3 %, one at 4 %, and one at $4\frac{1}{2}$ %. His total annual income from the three investments is $346. The first of these yields $44 per year more than the other two combined. If all the money were invested at $3\frac{1}{2}$ % he would receive $4 per year more than he does now. How much is invested at each rate?

5. For what value of a will the three planes represented by the equations given below have a line of intersection? Give the coordinates of three points on the line.

$$x + y + z = 6$$
$$y - z = 1$$
$$2x - 3y + az = 7$$

6. Three trucks were hauling concrete. The first day one truck hauled 4 loads, the second hauled 3 loads, and the third hauled 5 loads. The second day the trucks hauled 5, 4, and 4 loads respectively. The third day the same trucks hauled 3, 5, and 3 loads respectively. If the trucks hauled 78 cu. yds. the first day, 81 cu. yds.

the second day, and 69 cu. yds. the third day, find the capacity of each truck, assuming they were fully loaded on each trip.

7. Frank Nixon has a metal savings bank which registers the total amount deposited. Only pennies, nickels and dimes can be deposited. Frank knows that he has deposited one coin on each of 40 days. The bank shows a total deposit of $1.80. If Frank deposited as many pennies as both dimes and nickels, find the number of each.

8. A printing shop has three presses. One press operated 8 hours on Monday, 4 hours on Tuesday, and 2 hours on Wednesday. A second press operated 4 hours on Monday, 1 hour on Tuesday, and 5 hours on Wednesday. The third press operated 7 hours on Monday and 7 hours on Tuesday. Monday's output from the three presses was 1270 units, Tuesday's was 730 units, and Wednesday's was 550 units. What was the average output per hour for each press?

9. If A, B, C can do a piece of work in $2\frac{2}{3}$ days, A and B can do the work in $4\frac{4}{5}$ days, and C does twice as much work as A, at this rate, find the number of days in which each can do the work alone.

10. Three planes, A, B, C, working together can spray a certain cotton field in 2 hours. After they had worked together for one hour, plane C developed engine trouble, and planes A and B completed the job in one hour and 20 minutes more. The next day it was found necessary to respray the part sprayed by plane C. This was done by planes A and B in twenty minutes. How long would it take each plane to spray the entire field?

11. R, S, and T are the points of tangency of a triangle ABC circumscribed about a circle. If the sides of the triangle AB, BC, and AC are respectively 10, 8, and 7 units long,

find the lengths of the segments AS, SB, BT, TC, CR and AR.

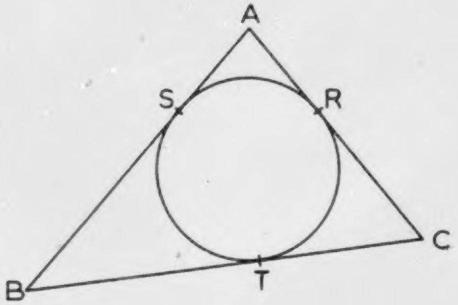


Figure 8-11a

12. If a parabola defined by the equation $y = ax^2 + bx + c$ passes through the points (-1,1), (3,1), (4,-4), find the values of the constants a, b and c.
13. If a parabola defined by $y = ax^2 + bx + c$ passes through the points (1,4), (-3,20), (-1,0), find the values of the constants a, b and c.
14. A local school gym entrance meter received half dollars from adults, quarters from high school pupils, and dimes from elementary school pupils. An attendant opened the box when the meter showed that 320 admissions had been deposited, giving a total of $76. He found there were twice as many dimes as quarters. Find the number of adults, high school pupils, and elementary school pupils who had paid admission.
15. The stopping distance of a car after the brakes are applied is given by the equation

$$s = \frac{1}{2}kt^2 + At + B$$

where

s = number of feet the car travels after the brakes are applied,

t = number of seconds the car is in motion after the brakes are applied.

If the following pairs of values were found for s and t, experimentally, find the values of the constants k, A, and B.

$$\begin{cases} s = 46 \\ t = 1 \end{cases}, \qquad \begin{cases} s = 84 \\ t = 2 \end{cases}, \qquad \begin{cases} s = 114 \\ t = 3. \end{cases}$$

16. Averages for a marking period in a certain mathematics class are based on scores made on a one-hour test, a short quiz, and a final examination. The scores made by Frank, Joyce, and Eunice, as well as their final averages, are shown in the following table.

	Test (T)	Quiz (Q)	Examination (E)	Final (A)
Frank	78	78	86	82
Joyce	78	98	74	80
Eunice	84	64	86	81

(a) Find values of w_1, w_2, w_3 that the instructor may have used to compute A if he used the formula

$$w_1T + w_2Q + w_3E = (w_1 + w_2 + w_3)A$$

to compute the final average, A.

(b) Can you find a triple of values for (w_1, w_2, w_3) whose sum is 1.

17. A firm sent a messenger to the post office to buy $10 worth of 7¢ air mail stamps, 4¢ stamps and 1¢ stamps. The directions given were to buy as many air mail and 4¢ stamps as possible, getting twice as many air mail stamps as 4¢ stamps, and buying one cent stamps with the change that remained after the air mail and four cent stamps had been purchased. How many of each kind of stamps will the messenger obtain?

18. After playing 18 holes of golf, a player reports his score as a certain number. His actual score is 1 stroke per hole greater than the number which he reports. If the number which he gave as his score and his actual score are averaged the resulting number is $\frac{1}{5}$ greater than par. A score of 2 over par is less than the number he reports by 1. What is par for the course, and what number does he report as his score?

*19. Find an equation for the plane containing the points (-1,0,0), (1,-1,0), (-1,3,2).

20. [NOTE: This problem should interest students who have studied chemistry.]

The problem of balancing chemical equations can be reduced to an easy algebraic process. We illustrate by several simple examples:

(a) Balance the equation for the following chemical reaction:

$$(\quad)FeS + (\quad)O_2 \longrightarrow (\quad)FeO + (\quad)SO_2$$

Insert the letters w, x, y, and z in the blanks and write down the equations resulting by equating the amounts of Fe in FeS and FeO

$$(w)\ FeS + (x)\ O_2 \longrightarrow (y)\ FeO + (z)\ SO_2$$

$$w = y \qquad w(Fe) = y(Fe)$$

Repeat this process for the sulfur and oxygen.

$$w(S) = z(S) \quad ; \quad w = z$$

$$x(2\ O) = y(O) + z(2\ O) \ ; \ 2x = y + 2z$$

$$\begin{aligned} w &= y \\ w &= z \\ 2x &= y + 2z \end{aligned}$$

Solve for x, y, and z in terms of w

$$\begin{aligned} y &= w \\ z &= w \\ x &= \frac{3}{2} w \end{aligned}$$

Choose w so that it is the smallest positive integer for which x, y, and z are also integers.

$$w = 2; \qquad x = 3$$
$$y = 2; \qquad z = 2$$

$$2\ FeS + 3\ O_2 \longrightarrow 2\ FeO + 2\ SO_2\ .$$

(b) Balance the equation for the following chemical reaction:

$(\quad)NH_3 + (\quad)O_2 \longrightarrow (\quad)H_2O + (\quad)NO_2$

$(a)NH_3 + (b)O_2 \longrightarrow (c)H_2O + (d)NO_2$

Nitrogen: $a = d$

Hydrogen: $3a = 2c$

Oxygen: $2b = c + 2d$

$d = a$

$c = \frac{3}{2}a$

$b = \frac{7}{4}a$

$\therefore$ a must be equal to 4

$b = 7; \quad c = 6; \quad d = 4$

$4\ NH_3 + 7\ O_2 \longrightarrow 6\ H_2O + 4\ NO_2$

Balance the equations for the following chemical reactions.

(a) $Ag + HNO_3 \longrightarrow AgNO_3 + NO + H_2O$

(b) $AuCl_3 + KI \longrightarrow AuCl + KCl + I_2$

(c) $HNO_3 + HI \longrightarrow NO + I_2 + H_2O$

(d) $MnO_2 + HCl \longrightarrow MnCl_2 + Cl_2 + H_2O$

(e) $Cr(OH)_3 + NaOH + H_2O_2 \longrightarrow Na_2CrO_4 + H_2O$

APPENDIX

*8-7. The Solution Set of a System of Two First Degree Equations in Three Variables. Graphic Approach.

In Section 8-3 we established the fact that every equation

$$Ax + By + Cz + D = 0$$

(in which A, B, and C are real coefficients not all zero) represents a plane. If we have two such first degree equations, they represent two planes that have one of three positions with respect to each other. The graphs of the two equations may intersect in a line, they may be parallel, or they may be the same plane. Our problem is to discuss the solution set of a system of two such equations. The most important case is the one in which the two planes intersect in a line. However, we will give an example to illustrate each of the three cases.

Example 1: The two planes intersect in a line. Find the solution set of the system

$$\begin{aligned} x + 2y + z - 5 &= 0, \\ x + z - 3 &= 0. \end{aligned} \tag{8-7a}$$

Solution: We use a method similar to one studied in Chapter 7. The complete solution set of the system (8-7a) may be obtained by studying the equivalent system obtained by combining either of the equations of (8-7a) with a combination

$$a(x + 2y + z - 5) + b(x + z - 3) = 0$$

of the equations of the system. By choosing $a = 1$, $b = -1$, we have

$$(x + 2y + z - 5) - (x + z - 3) = 0,$$

which reduces to

$$y = 1. \tag{8-7b}$$

Thus the line of intersection of the given planes, (8-7a), is also the line of intersection of the planes

$$\left\{\begin{aligned} x + 2y + z &= 5 \\ y &= 1 \end{aligned}\right. \quad \text{or} \quad \left\{\begin{aligned} x + z &= 3 \\ y &= 1. \end{aligned}\right.$$

(1) The easiest system to graph is the last one.

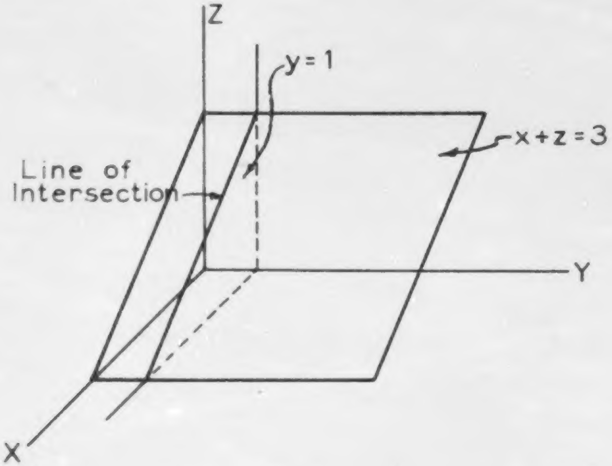

Z
y = 1
x + z = 3
Line of Intersection
Y
X

Figure 8-7a

[See also 8-7b and 8-7c.]

(2) Let us sketch the graph of the pair

$$x + 2y + z = 5,$$
$$y = 1.$$

The second plane is parallel to the XZ-plane and one unit to the right of it. Thus its trace in the XY-plane has a point of intersection with the XY-trace of the first plane; and its trace in the YZ-plane has a point of intersection with the YZ-trace of the first plane. Both these points have y = 1. They determine the line of intersection of the two planes. This line is parallel to the

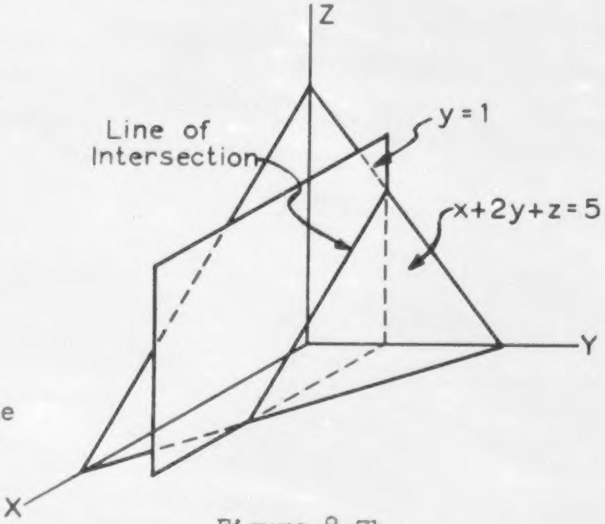

Z
Line of Intersection
y = 1
x+2y+z=5
Y
X

Figure 8-7b

[sec. 8-7]

XZ-plane.

(3) The third graph (Figure 8-7c) gives a sketch of the given planes

$$x + 2y + z = 5,$$

$$x + z = 3.$$

This graph is the most difficult to draw. The second plane has as its XY-trace the line

$$x = 3.$$

This intersects the XY-trace of the first plane, namely,

$$x + 2y = 5,$$

in the point $x = 3$, $y = 1$, $z = 0$. The traces of these two planes in the YZ-plane are

$$z = 3$$

$$2y + z = 5.$$

They intersect in the point

$$x = 0, \quad y = 1, \quad z = 3.$$

We see that the line of intersection of these two planes is the same line as the one we obtained in (1) and (2), and that it is parallel to the XZ-plane.

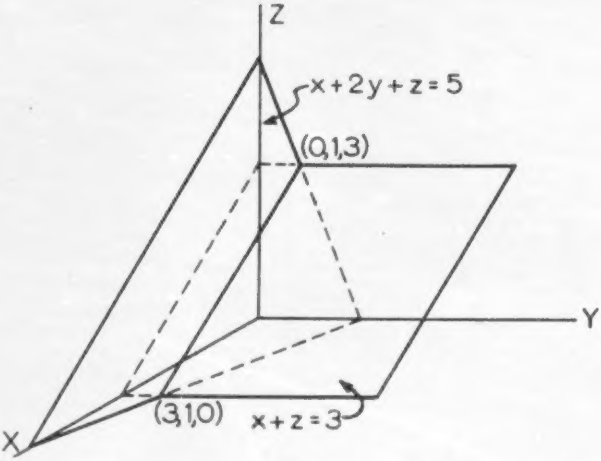


Figure 8-7c

Example 2. The two planes are parallel. Find the solution set of the system

$$x + 2y + z = 5,$$

$$x + 2y + z = 10.$$

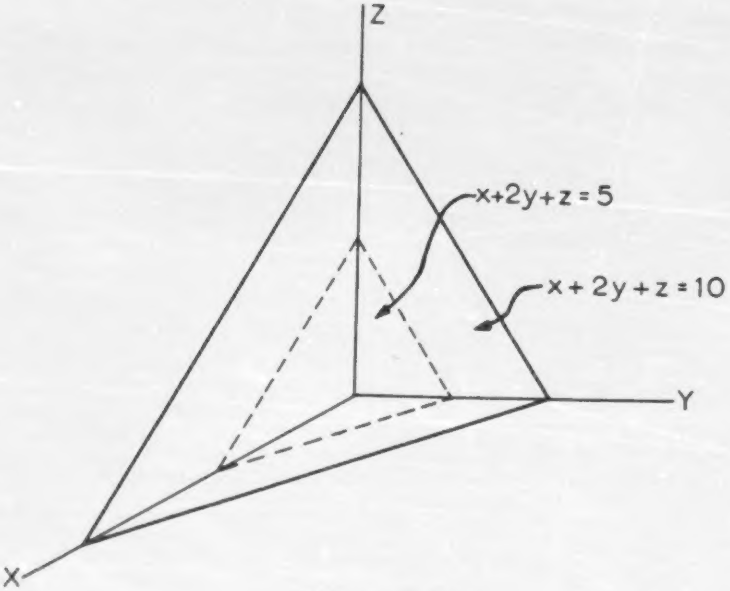


Figure 8-7d

Solution: By inspection, we can see that there is no number triple that satisfies both these equations. This is so because, for each number triple, the sum $(x + 2y + z)$ has a definite value that cannot be both 5 and 10.

The planes have no point in common; they are parallel. The system is inconsistent, since any triple (x, y, z) that satisfies one equation will not satisfy the other.

Example 3. The planes coincide. Find the solution set of the system

$$x + 2y + z - 5 = 0,$$

$$3x + 6y + 3z - 15 = 0.$$

Solution: By inspection, we can see that every number triple in the solution set of the first equation is also in the solution set of the second equation; and conversely. The given planes coincide. The system is dependent; the left member of the second equation is three times the left member of the first equation.

Exercises 8-7.

Determine which of the following pairs of equations represent straight lines. Sketch the graph in each case. When the planes intersect, indicate on the graph where the line of intersection lies.

1. $x - 2y + 5z = 10,$
   $z = 1.$
2. $x - 2y + 5z = 10,$
   $x = 4.$
3. $x - 2y + 5z = 10,$
   $y = -2.$
4. $x + y = 5,$
   $x = 7 + y.$
5. $x + y = 5,$
   $x + y + z = 10.$
6. $3y + z = 9,$
   $x + 4y = 4.$
7. $x + 4y = 4,$
   $z - x = 0.$
8. $3x + y - z = 2,$
   $2z = 6x + 2y - 4.$
9. $z - x = 0,$
   $3y + z = 9.$
10. $x = -2,$
    $z = 4.$
11. $x + 2y + z = 5,$
    $-x + 2y + z = 5.$
12. $x + 2y + z = 8,$
    $x - 2y = 0.$

---

*8-10. Equivalent Systems of Equations in Three Variables.

We give here a treatment of equivalent systems for first degree equations in three variables that is similar to the treatment developed for two equations in two variables in Chapter 7.

Recall the procedure used in Chapter 3 to study systems of first degree equations in two variables, as well as the methods used in Sections 8-7, 8-8, and 8-9 to study systems of first degree equations in three variables. We have been using the following operations which can always be performed upon the equations of a system to yield an equivalent system:

1. Two equations of the system may be interchanged.
2. An equation of the system may be multiplied by any number $k \neq 0$.
3. k times any equation of the system may be added to any other equation of the system.

Consider now the set of all equations that we can obtain from two given equations,

$$(8\text{-}10a) \qquad \begin{cases} x + 2y - z - 5 = 0 \\ x + y + z - 2 = 0, \end{cases}$$

by multiplying the first equation by a constant, a, and the second equation by a constant, b (where a and b are not both zero), and then adding the two equations. This procedure involves operations (2) and (3). Thus, we can represent all such equations by

$$(8\text{-}10b) \qquad a(x + 2y - z - 5) + b(x + y + z - 2) = 0$$

(a, b not both zero).

By definition, any solution of the system (8-10a) must reduce each of the expressions in the parentheses in (8-10b) to zero. It must therefore be a solution of (8-10b).

For example, if we take $a = 2$, $b = 1$, we obtain

$$(8\text{-}10c) \qquad \begin{array}{l} 2(x + 2y - z - 5) + 1(x + y + z - 2) = 0 \\ 3x + 5y - z - 12 = 0. \end{array}$$

Since this equation is of first degree, it represents a plane. Since it is satisfied by all the triples in the solution set of (8-10a), the plane passes through the line of intersection of the planes in (8-10a). Hence the equation (8-10c) represents a plane through the intersection of the planes in (8-10a).

Thus any two distinct planes formed by substituting values of a and b in (8-10b) determine the same line of intersection as the equations in (8-10a). The left members of the equations obtained from (8-10b) are called linear combinations of the left members of the equations in (8-10a). We have used this converse proposition in Sections 8-7, 8-8, and 8-9.

Example 1. Find the equations of 2 distinct planes through the line of intersection of the planes of the system

$$\begin{array}{l} y = 2 \\ z = 5. \end{array}$$

Sketch the graph.

Solution: The general equation of all planes through the intersection of the given planes is equation

(8-10d) $a(y - 2) + b(z - 5) = 0$ (a,b not both zero)

1. If we take $a = 1$, $b = 1$, we have

$$y - 2 + z - 5 = 0$$
$$y + z = 7.$$

The given plane $y = 2$ is parallel to the XZ-plane and 2 units to the right of it. The given plane $z = 5$ is parallel to the XY-plane and 5 units above it. These planes intersect in a line parallel to the X-axis. The new plane $y + z = 7$ has the following traces:

In the XY-plane where $z = 0$,

$$y = 7;$$

in the YZ-plane where $x = 0$,

$$y + z = 7;$$

in the XZ-plane where $y = 0$,

$$z = 7.$$

It is a plane parallel to the X-axis. (See Figure 8-10b.) Note that the YZ-trace, $y + z = 7$, passes through the point $y = 2$, $z = 5$ in the YZ-plane.

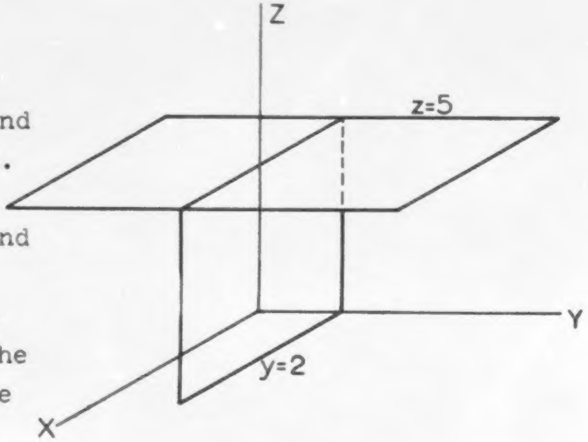


Figure 8-10a

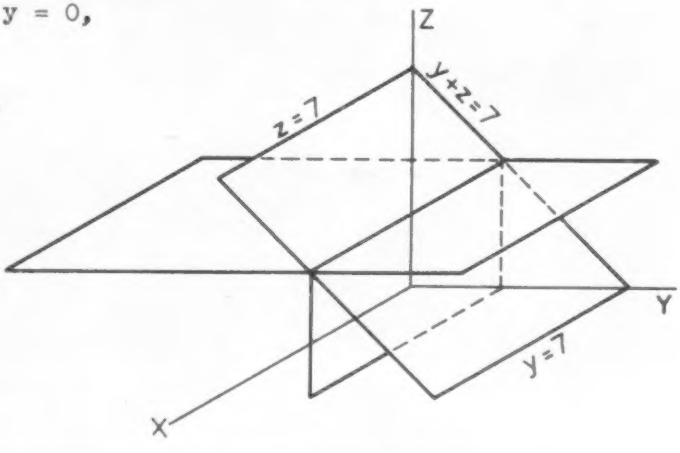


Figure 8-10b

2. If we take $a = 2$, $b = 2$ in equation (8-10d) we have

$$2(y - 2) + 2(z - 5) = 0,$$

$$2y + 2z - 14 = 0.$$

This plane coincides with the plane we have just studied,

$$y + z = 7.$$

This is because the a and b we have chosen are both twice the a and b chosen above.

3. If we take $a = 2$, $b = 1$, we have

$$2(y - 2) + (z - 5) = 0$$

$$2y + z - 9 = 0.$$

The traces of this plane are

$$z = 0\ , \quad y = \frac{9}{2};$$

$$x = 0\ , \quad 2y + z = 9;$$

$$y = 0\ , \quad z = 9.$$

This is another plane parallel to the X-axis. Notice again that the trace

$$2y + z = 9$$

passes through the point $(0, 2, 5)$. See Figure 8-10c.

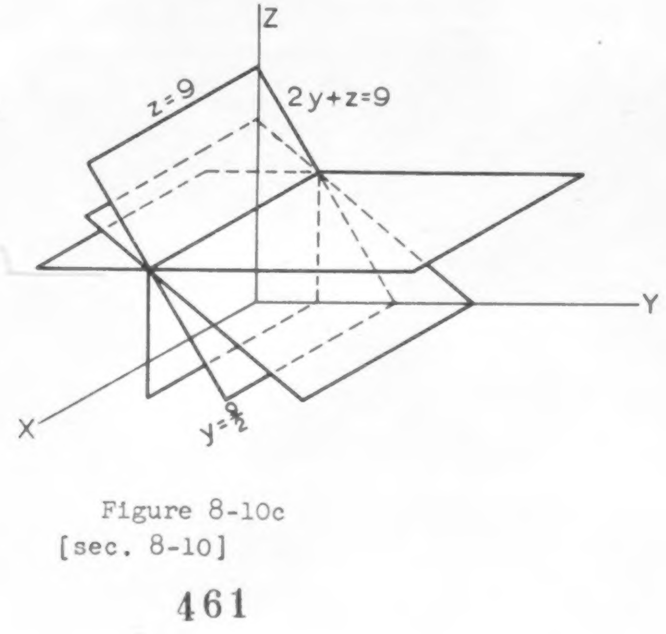


Figure 8-10c

[sec. 8-10]

Exercises 8-10.

1. Find an equation for a plane through the line of intersection of the planes in each of the following systems. By sketching the graph in each case, show that the plane represented by the equation you have found passes through the intersection of the given planes.

(a)	$x + 2 = 0,$ $z - 4 = 0.$	(b)	$y + 4 = 0,$ $z - 5 = 0.$

2. In each of the following problems, find an equation for the plane containing the given point and passing through the line of intersection of the given pair of planes.

(a)	$(1, 2, 1)$;	$x + 2y - 3z = 0,$ $x - y + z = 1.$
(b)	$(3, -1, 0)$;	$2y - 3z - 2 = 0,$ $x + y + z = 0.$
(c)	The origin ;	$x + z = 0,$ $2x - y + z - 8 = 0.$
(d)	$(2, 2, 1)$;	$2x - y + z - 3 = 0,$ $x - 3y + 4 = 0.$

*3. Prove that the planes represented by the equations

$$2x - y + 3z = 1$$
$$6x - 3y + 9z = 5$$

are parallel. Show that, for all values of a and b, both different from zero,

$$a(2x - y + 3z - 1) + b(6x - 3y + 9z - 5) = 0$$

represents a plane parallel to the given planes.

*4. Find an equation for the plane containing the point $(1, -1, 1)$ and passing through the line of intersection of the planes,

$$x + y - 3 = 0$$
$$z - 4 = 0.$$

Sketch the graph, showing the traces of the three planes, and show that the plane represented by the equation you have found passes through the intersection of the given planes.

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